

The Szegő kernel of a symplectic quotient

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1 Introduction.

The object of this paper is the relation between the Szegő kernel of an ample line bundle on a complex projective manifold, M , and the Szegő kernel of the induced polarization on the quotient of M by the holomorphic action of a compact Lie group, G .

Let M be an n -dimensional complex projective manifold and L an ample line bundle on it. Suppose a connected compact Lie group G acts on M as a group of holomorphic automorphisms, and that the action linearizes to L . Without loss of generality, we may then choose a G -invariant Kähler form Ω on M representing $c_1(L)$. We may also assume given a G -invariant Hermitian metric h on L such that the unique Hermitian connection on L compatible with the holomorphic structure has normalized curvature $-2\pi i\Omega$.

Let $\Phi : M \rightarrow \mathfrak{g}^*$, where \mathfrak{g} is the Lie algebra of G , be the moment map for the action (this is essentially equivalent to the assignment of a linearization). Suppose that $0 \in \mathfrak{g}^*$ lies in the image of Φ , that it is a regular value of Φ , and furthermore that G acts freely on $\Phi^{-1}(0)$. In this situation, the Hilbert-Mumford quotient for the action of the complexification \tilde{G} on M is nonsingular, and may be naturally identified with the symplectic reduction $M_0 = \Phi^{-1}(0)/G$. The latter, furthermore, inherits a naturally induced polarization L_0 , with an Hermitian metric h_0 and a Kähler form Ω_0 . The quotient structures L_0, h_0, Ω_0 are induced simply by descending the restrictions of L, h, Ω to $\Phi^{-1}(0)$ down to M_0 [GS2].

Let $H^0(M, L^{\otimes k})$ and $H^0(M_0, L_0^{\otimes k})$ denote the spaces of holomorphic sections of powers of L on M and of powers of L_0 on M_0 , respectively. Then G acts linearly on $H^0(M, L^{\otimes k})$, and in the given hypothesis by the theory of [GS2] there is for every integer $k \geq 0$ a natural isomorphism $H^0(M, L^{\otimes k})^G \cong$

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$H^0(M_0, L_0^{\otimes k})$. Here $H^0(M, L^{\otimes k})^G \subseteq H^0(M, L^{\otimes k})$ is the subspace of G -invariant holomorphic sections.

The given choices equip the vector spaces $H^0(M, L^{\otimes k})$ and $H^0(M_0, L_0^{\otimes k})$ with the following unitary structures $(,)$ and $(,)_0$, respectively. If $\sigma, \tau \in H^0(M, L^{\otimes k})$, then

$$(\sigma, \tau) =: \int_M h_p^{\otimes k}(\sigma(p), \tau(p)) \text{vol}_M(p),$$

where $\text{vol}_M =: \Omega^{\wedge n}$ is the volume form on M associated to the Kähler form Ω , and $h^{\otimes k}$ denotes the induced Hermitian metric on $L^{\otimes k}$. Similarly, if $\sigma_0, \tau_0 \in H^0(M_0, L_0^{\otimes k})$, then

$$(\sigma_0, \tau_0)_0 =: \int_{M_0} h_{0,p}^{\otimes k}(\sigma_0(p), \tau_0(p)) \text{vol}_{M_0}(p),$$

where now $\text{vol}_{M_0} =: \Omega_0^{\wedge(n-g)}$ is the volume form on M_0 associated to the Kähler form Ω_0 .

It is then natural to ask the following question: what are the asymptotic metric properties, with respect to these unitary structures, of the isomorphisms $r_k : H^0(M, L^{\otimes k})^G \rightarrow H^0(M_0, L_0^{\otimes k})$ as $k \rightarrow +\infty$? To simplify the exposition, we shall mostly leave the isomorphisms r_k implicit. From an analytic viewpoint, this question may be phrased as follows: how does the Szegő kernel of the pair (M, L) relate to the Szegő kernel of the quotient pair (M_0, L_0) ?

A natural measure of the difference between the two unitary structures is offered by the comparison of orthonormal basis. In this direction, recall the following basic fact from the theory of algebro-geometric Szegő kernels [Z]: given for every integer $k \geq 0$ an orthonormal basis $\{t_j^{(k)}\}$ of $H^0(M_0, L_0^{\otimes k})$, we have an asymptotic expansion, uniform in $p_0 \in M_0$,

$$\sum_j \left\| t_j^{(k)}(p_0) \right\|^2 \sim k^{n-g} + \sum_{l \geq 1} b_l(p_0) k^{n-g-l}, \quad (1)$$

where n is the complex dimension of M , g the real dimension of G , and the a_l 's are smooth functions on M_0 ; the right hand side of (1) is of course independent of the particular choice of the orthonormal basis $\{t_j^{(k)}\}$. In order to measure how far an orthonormal basis of $H^0(M, L^{\otimes k})^G$ is from being an orthonormal basis of $H^0(M_0, L_0^{\otimes k})$, we may then look for a similar asymptotic expansion for orthonormal basis of $H^0(M, L^{\otimes k})^G$, along $\Phi^{-1}(0) \subseteq M$.

The following is a special case of Theorem 1 of [P]: Given for every integer $k \geq 0$ an orthonormal basis $\{s_j^{(G,k)}\}$ of $H^0(M, L^{\otimes k})^G$, we have an asymptotic expansion, uniform in $p \in \Phi^{-1}(0)$:

$$\sum_j \left\| s_j^{(G,k)}(p) \right\|^2 \sim a_0(p) k^{n-g/2} + \sum_{l \geq 1} a_l(p) k^{n-g/2-l}, \quad (2)$$

where the a_l 's are smooth G -invariant functions on $\Phi^{-1}(0)$, and $a_0(p) > 0$ for every $p \in \Phi^{-1}(0)$. The same expression is rapidly decaying when $p \notin \Phi^{-1}(0)$. We are thus led to ask whether the r_k 's are asymptotically conformally isometric, by a conformal factor involving an appropriate power of k .

The following Theorem shows that this is not the case, unless the effective potential V_{eff} of the action is a constant. Recall that V_{eff} is the smooth function on M_0 whose value at $p_0 \in M_0$ is the volume of the fibre $\Phi^{-1}(p_0) \subseteq \Phi^{-1}(0)$ in the restricted metric. This is an important attribute of the action, playing a crucial role in the study of the Kähler structure of the quotient [BG]. It may be viewed as the G -invariant function on $\Phi^{-1}(0)$ associating to each $p \in \Phi^{-1}(0)$ the volume of the fibre $G \cdot p$.

Theorem 1. *Suppose as above that $0 \in \mathfrak{g}^*$ is a regular value of the moment map, and that G acts freely on $\Phi^{-1}(0)$. Let n be the complex dimension of M and g be the real dimension of G . For every integer $k \geq 0$ let $\{s_j^{(G,k)}\}$ be an orthonormal basis of the space $H^0(M, L^{\otimes k})^G$ of G -invariant holomorphic sections of $L^{\otimes k}$. Then there is an asymptotic expansion, uniform in $p \in \Phi^{-1}(0)$,*

$$\sum_j \left\| s_j^{(G,k)}(p) \right\|^2 \sim \frac{1}{V_{\text{eff}}(p)} k^{n-g/2} + \sum_{l \geq 1} a_l(p) k^{n-g/2-l}. \quad (3)$$

Let now the weights $\{\omega\}$ label the finite dimensional irreducible representations of the connected compact Lie group G ; let V_ω be the representation associated to the weight ω . Given the linear action of G on $H^0(M, L^{\otimes k})$, there is a G -equivariant direct sum decomposition

$$H^0(M, L^{\otimes k}) \cong \bigoplus_{\omega} H^0(M, L^{\otimes k})_{\omega}$$

of $H^0(M, L^{\otimes k})$ over the finite dimensional irreducible representations of G ; for every weight ω the subspace $H^0(M, L^{\otimes k})_{\omega}$ is G -equivariantly isomorphic to a direct sum of copies of V_ω . Pairing Theorem 1 above with Theorem 1 of [P], we immediately get:

Corollary 1. *In the hypothesis of Theorem 1, let ω be the weight corresponding to a finite dimensional irreducible representation of G . For every integer $k \geq 0$ let $\{s_j^{(\omega,k)}\}$ be an orthonormal basis of the subspace $H^0(M, L^{\otimes k})_\omega \subseteq H^0(M, L^{\otimes k})$. Then there is an asymptotic expansion, uniform in $p \in \Phi^{-1}(0)$,*

$$\sum_j \left\| s_j^{(\omega,k)}(p) \right\|^2 \sim \frac{\dim(V_\omega)^2}{V_{\text{eff}}(p)} k^{n-g/2} + \sum_{l \geq 1} a_{l,\omega}(p) k^{n-g/2-l}. \quad (4)$$

The asymptotic expansion in Theorem 1 is a straightforward consequence of the following analytic core result, which we state here rather loosely and that will be spelled out more precisely in the course of the paper:

Key Result: The Szegő kernel of the triple (M, L, h) descends, in an appropriate sense, to an elliptic Toeplitz operator on the circle bundle of the quotient triple (M_0, L_0, h_0) , and after a suitable renormalization the symbol of this Toeplitz operator is (proportional to) the effective volume of the action.

The techniques in this paper are based in a general sense on the microlocal theory of the Szegő kernel [BS], and more specifically on its formulation based on Fourier-Hermite distributions developed in [BG]; in this framework one has a good control of the functorial behaviour of Toeplitz operators and their symbols under geometric operations like restriction and push-forward. The general strategy used here was also inspired by the approach of Shiffman, Tate and Zelditch to Szegő kernels on toric varieties [STZ], especially by their philosophy of restricting the Szegő kernel of an ambient projective space to a projective submanifold. In fact, here we shall first restrict the Szegő kernel of (M, L, h) to the locus where the moment map vanishes, and then push it forward by the G -action.

2 Metalinear and metaplectic preliminaries.

We shall make use of the notions of metalinear manifold and half-form [GS1]. If Z is a manifold, we shall denote by $|\wedge|(Z)$ the line bundle of densities on Z , and by $|\Omega|(Z)$ its space of smooth global sections, which we simply call densities on Z . If Z is a metalinear manifold, we shall denote by $\wedge^{1/2}(Z)$ the line bundle of metalinear forms on Z , and by $\Omega^{1/2}(Z)$ its space of smooth global sections, which we simply call metalinear forms on Z . An orientation on a manifold induces a metalinear structure on it, and a volume form induces a nowhere vanishing half-form, its *square root*. If $\mu, \nu \in \Omega^{1/2}(Z)$, the product

$\mu \cdot \bar{\nu}$ is a density on X . The space of square integrable half-forms on a metalinear manifold has a natural Hilbert structure, given by

$$\langle \mu, \nu \rangle = \int_Z \mu \cdot \bar{\nu} \quad (\mu, \nu \in \Omega^{1/2}(Z)).$$

There is a more general notion of metalinear vector bundle, a metalinear manifold being a manifold with a metalinear tangent bundle. To a metalinear structure on a rank- r vector bundle E there is associated a line bundle $\bigwedge^{1/2}(E)$ satisfying

$$\bigwedge^{1/2}(E) \otimes \bigwedge^{1/2}(E) \cong \bigwedge^r(E^*),$$

where the latter is the top exterior power of the dual E^* . The space of smooth global sections of $\bigwedge^{1/2}(E)$ will be denoted by $\Omega_E^{1/2}$. Thus, if Z is a metalinear manifold then $\bigwedge^{1/2}(Z) = \bigwedge^{1/2}(TZ)$ and $\Omega^{1/2}(Z) = \Omega_{TZ}^{1/2}$. If E is oriented, it is metalinear, and if $0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$ is a short exact sequence of vector bundles on a manifold, then any of the three vector bundles is metalinear if the other two are, and $\bigwedge^{1/2}(E) \cong \bigwedge^{1/2}(A) \otimes \bigwedge^{1/2}(B)$. A nowhere vanishing smooth section of $\bigwedge^r(E^*)$ induces a metalinear structure and a nowhere vanishing section of $\bigwedge_E^{1/2}$, $\sqrt{\sigma} \in \Omega_E^{1/2}$.

In the following, we shall be making various natural choices of a nowhere vanishing section of $\bigwedge^r(E^*)$ for some vector bundles in our picture (r being the rank of E); we shall generally denote this choice by vol_E and the associated nowhere vanishing section of $\bigwedge_E^{1/2}(E)$ by $\text{vol}_E^{(1/2)} = \sqrt{\text{vol}_E}$.

Let P and Q be metalinear manifolds. A morphism of metalinear manifolds from P to Q is the assignment of a smooth map $f : P \rightarrow Q$ and a morphism of vector bundles $\tilde{f} : f^* \left(\bigwedge^{1/2}(Z) \right) \rightarrow \bigwedge^{1/2}(P)$. Let $f : P \rightarrow Q$ be any smooth map. Let

$$\Gamma_f = \{ (p, -(d_p f)^t(\eta)), (f(p), \eta) : p \in P, \eta \in T_{f(p)}^* Q \}$$

be the conormal bundle to the graph of f . Then Γ_f is clearly a Lagrangian submanifold of $T^*P \times T^*Q$ and a metalinear manifold. Furthermore, giving f the structure of a morphism of metalinear manifolds is equivalent to assigning an appropriate half-form on Γ_f [GS1].

Let us suppose, in particular, that Q is an oriented Riemannian manifold. We shall denote by η_Q the half-form taking value one on oriented orthonormal frames of TQ , and call it the *canonical* half-form of Q . If P is also an oriented Riemannian manifold, we can then make any smooth map $f : P \rightarrow Q$ into a morphism of metalinear manifolds by setting $\tilde{f}(\eta_Q) = \eta_P$. Let

$\Gamma_f \subseteq T^*(P \times Q) \setminus \{0\}$ be the conormal bundle to the graph of f , with projections $q_1 : \Gamma_f \rightarrow P$ and $q_2 : \Gamma_f \rightarrow Q$. Given the exact sequence

$$0 \longrightarrow q_2^*(T^*Q) \rightarrow T(\Gamma_f) \longrightarrow q_1^*(TP) \longrightarrow 0,$$

the half-form on Γ_f associated to \tilde{f} is

$$\text{vol}_{\Gamma_f}^{1/2} =: q_2^*(\eta_Q^{-1}) \otimes q_1^*(\eta_P). \quad (5)$$

If a morphism of metlinear manifolds is given, passing to global sections we can define the pull-back of a metlinear half-form on Q to a metlinear half-form on P ; we shall simply denote this by $f^* : \Omega^{1/2}(Q) \rightarrow \Omega^{1/2}(P)$. If, in addition, f is a proper submersion, we can also define a push-forward operation $f_* : \Omega^{1/2}(P) \rightarrow \Omega^{1/2}(Q)$. The microlocal theory for these operations on Fourier and Fourier-Hermite generalized half-forms has been developed in [GS1] and [BG].

The proof of the following Lemma is left to the reader:

Lemma 1. *Let P, Q be oriented Riemannian manifolds, and let $f : P \rightarrow Q$ be a proper submersion. Let us make f into a morphism of metlinear manifolds by setting $\tilde{f}^*(\eta_Q) =: \eta_P$. For $q \in Q$, let $v(q)$ denote the volume of the fibre $f^{-1}(q)$ in the induced metric structure. Then the associated push-forward operation $f_* : \Omega^{1/2}(P) \rightarrow \Omega^{1/2}(Q)$ satisfies*

$$f_*(\eta_P) = v \cdot \eta_Q.$$

We shall now review some basic facts and notation from metplectic geometry. Let $\text{Sp}(\ell)$ be the group of $2\ell \times 2\ell$ (real) symplectic matrices, and for $1 \leq k \leq \ell$ define:

$$\text{Sp}(k, \ell) = \{A = [a_{ij}] \in \text{Sp}(\ell) : a_{ij} = 0 \text{ if } 1 \leq j \leq k, i > k\}.$$

There is an obvious (surjective) morphism of Lie groups

$$\text{Sp}(k, \ell) \xrightarrow{\varpi} \text{GL}(k) \times \text{Sp}(\ell - k). \quad (6)$$

The following lemma is left to the reader:

Lemma 2. *We have:*

$$\begin{aligned} \ker(\varpi) = & \left\{ \begin{pmatrix} I_k & B & A_{11} & A_{12} \\ 0 & I_{\ell-k} & A_{12}^t & A_{22} \\ 0 & 0 & I_k & 0 \\ 0 & 0 & -B^t & I_{\ell-k} \end{pmatrix} : B \in M_{k, \ell-k}(\mathbb{R}), A_{12} \in M_{k, \ell-k}(\mathbb{R}), \right. \\ & \left. \text{and } A_{11} \in M_k(\mathbb{R}), A_{22} \in M_{\ell-k}(\mathbb{R}) \text{ are both symmetric} \right\}. \end{aligned}$$

Let $T =: \ker(\varpi) \subseteq \mathrm{Sp}(k, \ell)$. Then T is a contractible, hence simply connected, normal subgroup of $\mathrm{Sp}(k, \ell)$.

Similarly, let us define

$$\mathrm{Sp}(k, \ell)_+ = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1k} & a_{1k+1} & \cdots & a_{1\ell} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{k1} & \cdots & a_{kk} & a_{kk+1} & \cdots & a_{k\ell} \\ 0 & \cdots & 0 & a_{k+1k+1} & \cdots & a_{k+1\ell} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{\ell k+1} & \cdots & a_{\ell\ell} \end{pmatrix} \in \mathrm{Sp}(k, \ell) \right. \\ \left. : \det \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \cdots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix} > 0 \right\}.$$

By restriction of ϖ , we obtain a surjective morphism of Lie groups, with kernel $T \subseteq \mathrm{Sp}(k, \ell)_+$:

$$\mathrm{Sp}_+(k, \ell) \xrightarrow{\varpi_+} \mathrm{GL}_+(k) \times \mathrm{Sp}(\ell - k). \quad (7)$$

Therefore, $\mathrm{Sp}_+(k, \ell)$ is a connected subgroup of $\mathrm{Sp}(\ell)$.

Let now $s : \mathrm{Mp}(\ell) \rightarrow \mathrm{Sp}(\ell)$ be the metaplectic double cover. Now $\mathrm{GL}(\ell)$ sits naturally in $\mathrm{Sp}(\ell)$ in the standard manner, and

Lemma 3. $s^{-1}(\mathrm{GL}(\ell)) \cong \mathrm{ML}(\ell)$.

Let us next define $\mathrm{Mp}(k, \ell) =: s^{-1}(\mathrm{Sp}(k, \ell))$, and $\mathrm{Mp}_+(k, \ell) =: s^{-1}(\mathrm{Sp}_+(k, \ell))$.

Lemma 4. $\mathrm{Mp}_+(k, \ell)$ is a connected Lie group.

Proof. Let us consider the injective morphism of Lie groups $\psi : \mathrm{Sp}(\ell - k) \rightarrow \mathrm{Sp}(\ell)$ given by

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \mapsto \begin{pmatrix} I_k & 0 & 0 & 0 \\ 0 & A_1 & 0 & A_2 \\ 0 & 0 & I_k & 0 \\ 0 & A_3 & 0 & A_4 \end{pmatrix}. \quad (8)$$

Clearly, $\psi(\mathrm{Sp}(\ell - k)) \subseteq \mathrm{Sp}(k, \ell)_+$, and ψ induces an isomorphism of homotopy groups $\psi_* : \pi_1(\mathrm{Sp}(\ell - k)) \cong \pi_1(\mathrm{Sp}(\ell)) \cong \mathbb{Z}$. Thus,

$$s^{-1}[\psi(\mathrm{Sp}(\ell - k))] \cong \mathrm{Mp}(\ell - k)$$

is the unique connected 2:1 cover of $\mathrm{Sp}(\ell - k)$. Hence, on the one hand, $\mathrm{Mp}(\ell - k)$ is contained in the identity component of $\mathrm{Mp}(k, \ell)_+$, and on the

other if there was another connected component of $\text{Mp}(k, \ell)_+$ then points in $\psi(\text{Sp}(\ell - k))$ would have more than two inverse images in $\text{Mp}(\ell)$, absurd.

Let us now consider the surjective homomorphism $\gamma : \text{Sp}(k, \ell)_+ \rightarrow \text{Sp}(\ell - k)$ given by the composition of ϖ with projection onto the second factor. The following Lemma is left to the reader:

Lemma 5. *We have:*

$$\begin{aligned} \ker(\gamma) = & \left\{ \begin{pmatrix} A & B & A_{11} & A_{12} \\ 0 & I_{\ell-k} & A_{12}^t & A_{22} \\ 0 & 0 & (A^{-1})^t & 0 \\ 0 & 0 & -(A^{-1}B)^t & I_{\ell-k} \end{pmatrix} : \right. \\ & A \in \text{GL}_+(k), B \in M_{k, \ell-k}(\mathbb{R}), A_{12} \in M_{k, \ell-k}(\mathbb{R}), \\ & \left. \text{and } A_{11} \in M_k(\mathbb{R}), A_{22} \in M_{\ell-k}(\mathbb{R}) \text{ are both symmetric} \right\}. \end{aligned}$$

Thus, $Z =: \ker(\gamma)$ retracts to $\text{GL}_+(k) \subseteq \text{Sp}_+(k, \ell)$. Now $s^{-1}(\text{GL}(\ell)) \subseteq \text{Mp}(\ell)$ is simply the metaleinear group $\text{ML}(\ell)$, and consists of four connected components. Therefore, the inverse image $s^{-1}(Z) \subseteq \text{Mp}_+(k, \ell)$ is the union of two connected components, each mapping isomorphically onto Z . We shall also denote by Z the identity component of $s^{-1}(Z)$.

Lemma 6. $\text{Mp}_+(k, \ell)/Z \cong \text{Mp}(\ell - k)$.

Proof. The commutative diagram

$$\begin{array}{ccc} \text{Mp}_+(k, \ell) & \longrightarrow & \text{Mp}_+(k, \ell)/Z \\ \downarrow & & \downarrow \\ \text{Sp}_+(k, \ell) & \longrightarrow & \text{Sp}(\ell - k) = \text{Sp}_+(k, \ell)/Z \end{array}$$

exhibits $\text{Mp}_+(k, \ell)/Z$ as a connected double cover of $\text{Sp}(\ell - k)$.

$\text{GL}(k)$ and $\text{Sp}(\ell - k)$ sit naturally in $\text{Sp}(k, \ell)$, namely as the commuting Lie subgroups of all matrices of the form

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & I_{\ell-k} & 0 & 0 \\ 0 & 0 & (A^t)^{-1} & 0 \\ 0 & 0 & 0 & I_{\ell-k} \end{pmatrix} \quad (A \in \text{GL}(k))$$

and

$$\begin{pmatrix} I_k & 0 & 0 & 0 \\ 0 & R & 0 & S \\ 0 & 0 & I_k & 0 \\ 0 & T & 0 & U \end{pmatrix} \quad \left(\begin{pmatrix} R & S \\ T & U \end{pmatrix} \in \text{Sp}(\ell - k) \right),$$

respectively. We have therefore an injective morphism of Lie groups

$$o = (\zeta, \chi) : \mathrm{GL}(k) \times \mathrm{Sp}(\ell - k) \rightarrow \mathrm{Sp}(k, \ell) \subseteq \mathrm{Sp}(\ell).$$

The inverse images $s^{-1}(\mathrm{GL}(k))$ and $s^{-1}(\mathrm{Sp}(\ell - k))$ are isomorphic, respectively, to $\mathrm{ML}(k)$ and $\mathrm{Mp}(\ell - k)$. Thus, there is a commutative diagram of Lie group homomorphisms

$$\begin{array}{ccccc} \mathrm{ML}(k) & \xrightarrow{\tilde{\xi}} & \mathrm{Mp}(k, \ell) & \xleftarrow{\tilde{\chi}} & \mathrm{Mp}(\ell - k) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{GL}(k) & \xrightarrow{\xi} & \mathrm{Sp}(k, \ell) & \xleftarrow{\chi} & \mathrm{Sp}(\ell - k). \end{array}$$

Taking products, we have a smooth map

$$\tilde{o} = (\tilde{\xi}, \tilde{\chi}) : \mathrm{ML}(k) \times \mathrm{Mp}(\ell - k) \longrightarrow \mathrm{Mp}(k, \ell) \subseteq \mathrm{Mp}(\ell), \quad (g, h) \mapsto \tilde{\xi}(g) \cdot \tilde{\chi}(h).$$

Lemma 7. *\tilde{o} is a Lie group homomorphism.*

Proof. Given that $\tilde{\xi}$ and $\tilde{\chi}$ are morphisms, it suffices to show that

$$\tilde{\xi}(g) \cdot \tilde{\chi}(h) \cdot \tilde{\xi}(g)^{-1} \cdot \tilde{\chi}(h)^{-1} = e, \quad \forall (g, h) \in \mathrm{ML}(k) \times \mathrm{Mp}(\ell - k).$$

Given that o is a morphism, these commutators all lie in $\ker(s)$, which is finite of order 2. Given that each of $\tilde{\xi}$ and $\tilde{\chi}$ is a morphism, they all equal e if at least one of g and h is the identity. By connectedness of $\mathrm{Mp}(\ell - k)$, the claim follows.

The composition

$$\varpi \circ s|_{\mathrm{Mp}(k, \ell)} \circ \tilde{o} : \mathrm{ML}(k) \times \mathrm{Mp}(\ell - k) \longrightarrow \mathrm{GL}(k) \times \mathrm{Sp}(\ell - k)$$

is clearly the product of the double covers $\mathrm{ML}(k) \rightarrow \mathrm{SL}(k)$ and $\mathrm{Mp}(\ell - k) \rightarrow \mathrm{Sp}(\ell - k)$, and is thus a 4:1 covering.

Being contractible, T lifts isomorphically to a subgroup of $\mathrm{Mp}(\ell)$, that we still denote $T \triangleleft \mathrm{Mp}(k, \ell)$. Let us define the quotient group $\mathrm{Mp}(k|\ell) = \mathrm{Mp}(k, \ell)/T$. Let $\theta : \mathrm{Mp}(k, \ell) \rightarrow \mathrm{Mp}(k|\ell)$ be the projection. On the upshot, we have:

Lemma 8. *There is a commutative diagram of morphisms of Lie groups*

$$\begin{array}{ccccc}
& & \kappa_1 & & \\
& & \longrightarrow & & \\
& \text{Mp}(k : \ell) & & \text{ML}(k) \times \text{Mp}(\ell - k) & \\
& & & & \\
\kappa_2 & \downarrow & \tilde{\sigma} & \downarrow & \mu \\
& & \swarrow \theta & & \\
& \text{Mp}(k, \ell) & \longrightarrow & \text{Mp}(k|\ell) & \\
& & & & \\
s & \downarrow & & \downarrow & \nu \\
& & \varpi & & \\
& \text{Sp}(k, \ell) & \longrightarrow & \text{GL}(k) \times \text{Sp}(\ell - k) &
\end{array} \tag{9}$$

where the upper square is a fibre product diagram, μ and ν are 2:1 coverings, and $\nu \circ \mu$ is the product of the 2:1 coverings $\text{ML}(k) \rightarrow \text{GL}(k)$ and $\text{Mp}(\ell - k) \rightarrow \text{Sp}(\ell - k)$.

Recall that for every integer $l \geq 1$ the metalelinear group $\text{ML}(l)$ has four connected component. If $r : \text{ML}(l) \rightarrow \text{GL}(l)$ is the projection, the composition $\det \circ r : \text{ML}(l) \rightarrow \mathbb{R}$ admits a square root, that we shall denote by $\sqrt{\det} : \text{ML}(l) \rightarrow \mathbb{C}$. We shall denote the four connected components of $\text{ML}(l)$ by $\text{ML}(l)_1$, $\text{ML}(l)_{-1}$, $\text{ML}(l)_i$, $\text{ML}(l)_{-i}$, meaning that $\sqrt{\det} > 0$ on $\text{ML}(l)_1$, that $\sqrt{\det}$ is positive imaginary on $\text{ML}(l)_i$, and so forth. Clearly, $\text{ML}(l)_1$ is the identity component, and $r^{-1}(I_\ell) = \{e, g_l\}$ where e is the identity and $g_l \in \text{ML}(l)_{-1}$. Given the standard inclusion $\text{ML}(l) \hookrightarrow \text{Mp}(l)$, we shall view g_l as sitting in $\text{Mp}(l)$.

Lemma 9. $\ker(\mu) = \ker(\tilde{\sigma}) = \{e, (g_k, g_{\ell-k})\}$, where $e \in \text{ML}(k) \times \text{Mp}(\ell - k)$ is the unit.

Proof. Since μ is a 2:1 cover, it suffices to show that $(g_k, g_{\ell-k}) \in \ker(\tilde{\sigma})$. Since $s \circ \tilde{\sigma}((g_k, g_{\ell-k})) \in T = \ker(\varpi)$, and T has no elements of order 2, $s \circ \tilde{\sigma}((g_k, g_{\ell-k}))$ is the identity. Given that $\tilde{\sigma}((g_k, g_{\ell-k}))$ lies in $\text{ML}(\ell)$, it suffices to notice that $\sqrt{\det} \circ \tilde{\sigma}((g_k, g_{\ell-k})) = (-1) \cdot (-1) = 1$.

Corollary 2. *By restriction of $\tilde{\sigma}$, we have an injective morphism of Lie groups $\text{GL}_+(k) \times \text{Mp}(\ell - k) \hookrightarrow \text{Mp}(k, \ell) \subseteq \text{Mp}(\ell)$.*

Let us identify the unitary group $\text{U}(\ell)$ with the maximal compact subgroup of $\text{Sp}(\ell)$ consisting of all $2\ell \times 2\ell$ symplectic matrices of the form

$$\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \text{ such that } X + iY \in \text{U}(\ell).$$

Notice that $T \cap U(\ell) = \{I_{2\ell}\}$ and $Z \cap U(\ell) \cong O(k)$.

Similarly, let us set $U(k, \ell) = \text{Sp}(k, \ell) \cap U(\ell)$ and $U_+(k, \ell) = \text{Sp}_+(k, \ell) \cap U(\ell)$, so that

$$\begin{aligned} U(k, \ell) &= \left\{ \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & R & 0 & S \\ 0 & 0 & A & 0 \\ 0 & -S & 0 & R \end{pmatrix} : A \in O(k), \begin{pmatrix} R & -S \\ S & R \end{pmatrix} \in U(\ell - k) \right\} \\ &\cong O(k) \times U(\ell - k), \end{aligned}$$

and

$$U_+(k, \ell) \cong SO(k) \times U(\ell - k)$$

is the identity component of $U(k, \ell)$.

Let $\text{MU}(\ell) = s^{-1}(U(\ell)) \subseteq \text{Mp}(\ell)$. This is a connected compact subgroup of $\text{Mp}(\ell)$. Similarly, let $\text{MU}(k, \ell) = s^{-1}(U(k, \ell)) \subseteq \text{Mp}(k, \ell)$. If we let $\text{MO}(k) = s^{-1}(O(k))$, the vertical maps in the commutative diagram (9) both reduce to the composition of double covers

$$\text{MO}(k) \times \text{MU}(\ell - k) \longrightarrow \text{MU}(k, \ell) \longrightarrow U(k, \ell),$$

and the analogue of Corollary 2 is

Corollary 3. *By restriction of \tilde{o} , we have an injective morphism of Lie groups $SO(k) \times \text{MU}(\ell - k) \hookrightarrow \text{MU}_+(k, \ell) \subseteq \text{MU}(\ell)$.*

For any $l \geq 1$, let $\mathcal{S}(\mathbb{R}^l)$ be the space of smooth complex valued rapidly decaying functions on \mathbb{R}^l , endowed with the bilinear pairing

$$(f, g) =: \int_{\mathbb{R}^l} f(x) g(x) dx \quad (f, g \in \mathcal{S}(\mathbb{R}^l)),$$

and the L^2 -Hermitian product $(f, g)_h =: (f, \bar{g})$. The metaplectic group acts unitarily on $(\mathcal{S}(\mathbb{R}^l), (\cdot, \cdot)_h)$ under the Segal-Shale-Weyl representation,

$$v_{\text{SSW}} : \text{Mp}(l) \longrightarrow U(\mathcal{S}(\mathbb{R}^l)),$$

which plays a crucial role in the symbolic calculus of Fourier-Hermite distributions.

Given the inclusion of Lie groups of Corollary 2 (or Corollary 3), we may restrict the Segal-Shale-Weyl representation of $\text{Mp}(\ell)$ to $\text{GL}_+(k) \times \text{Mp}(\ell - k)$. On the other hand, $\text{GL}_+(k) \times \text{MU}(\ell - k)$ acts on $\mathcal{S}(\mathbb{R}^{\ell - k}) = \mathbb{C} \otimes_{\mathbb{C}} \mathcal{S}(\mathbb{R}^{\ell - k})$ by the tensor product $\sqrt{\det} \otimes v_{\text{SSW}}$ of the character $\sqrt{\det}$ on $\text{GL}_+(k) \subseteq \text{ML}(k)$ and the Segal-Shale-Weyl representation of $\text{Mp}(\ell - k)$. Using the description of the metaplectic representation in [GS1] and [BG], one can check that

Lemma 10. *The restriction map $\mathcal{S}(\mathbb{R}^\ell) \longrightarrow \mathcal{S}(\mathbb{R}^{\ell-k})$,*

$$f(x_1, \dots, x_\ell) \mapsto f_{\text{res}}(x_{k+1}, \dots, x_\ell) =: f(0, \dots, 0, x_{k+1}, \dots, x_\ell) \quad (f \in \mathcal{S}(\mathbb{R}^\ell))$$

is equivariant, that is, it is a morphism of $\text{GL}_+(k) \times \text{MU}(\ell - k)$ -modules.

In the following, we shall use the concepts of metaplectic manifold and metaplectic vector bundle, a manifold being metaplectic if its tangent bundle is. If (E, Ω_E) is a symplectic vector bundle of rank 2ℓ over a manifold N , we shall denote by $\text{Bp}(E) \rightarrow N$ the principal $\text{Sp}(\ell)$ -bundle of all symplectic frames in E . If the symplectic vector bundle (E, Ω_E) is metaplectic, we shall denote by $\widetilde{\text{Bp}}(E)$ the corresponding principal $\text{Mp}(\ell)$ -bundle.

In particular, if E admits a Lagrangian subbundle $L \subseteq E$, then E is symplectically equivalent to the vector bundle $L \oplus L^*$, with its standard symplectic structure. The structure group of E then reduces to $\text{GL}(\ell) \subseteq \text{Sp}(\ell)$, and E is metaplectic if and only if L is metalinear. In particular, a cotangent bundle T^*M is a metaplectic manifold if and only the base manifold M is metalinear, and furthermore any Lagrangian submanifold of a metaplectic manifold is metalinear.

Let us now assume, more generally, that $S \subseteq E$ is a rank- k isotropic vector subbundle. The subbundle $\text{Bp}(S, E) \subseteq \text{Bp}(E)$ consisting of all symplectic basis whose first k vectors lie in S is a principal $\text{Sp}(k, \ell)$ -bundle over N . Let $\text{BL}(S)$ denote the principal $\text{GL}(k)$ -bundle over N consisting of all linear frames in S . Let $S^{\perp\Omega} \subseteq E$ be the symplectic annihilator of S . We have $\text{BL}(S) \times \text{Bp}(S^{\perp\Omega}/S) = \text{Bp}(S, E)/T$, the projection being the bundle map

$$(e_1, \dots, e_\ell, f_1, \dots, f_\ell) \mapsto ((e_1, \dots, e_\ell), ([e_{k+1}], \dots, [e_\ell], [f_{k+1}], \dots, [f_\ell])), \quad (10)$$

obviously equivariant with respect to the morphism of Lie groups (6).

Now suppose, in addition, that S is orientable. Let $\text{BL}_+(S) \subseteq \text{BL}(S)$ be the principal $\text{GL}_+(k)$ -bundle of all oriented frames in S . Let $\text{Bp}_+(S, E) \subseteq \text{Bp}(S, E)$ be the subbundle consisting of all symplectic basis whose first k vectors form an *oriented* basis of S . Then $\text{Bp}_+(S, E)$ is a principal $\text{Sp}(k, \ell)_+$ -bundle over N , and the projection $\text{Bp}_+(S, E) \longrightarrow \text{BL}_+(S) \times \text{Bp}(S^{\perp\Omega}/S)$ is equivariant with respect to the morphism of Lie groups ϖ_+ in (7).

Proposition 1. *Let (E, Ω_E) be a symplectic vector bundle, and let $I \subseteq E$ be an oriented rank- k isotropic vector subbundle. Let $I^{\perp\Omega} \subseteq E$ denote the symplectic annihilator of S in E . Then there is a natural bijection between the set of equivalence classes of metaplectic structures on the symplectic vector bundle $N_I =: I^{\perp\Omega}/I$ and the set of equivalence classes of metaplectic structures on E .*

Proof. In one direction, suppose given a metaplectic structure on E , that we describe by the following equivariant commutative diagram of principal bundles and double covers:

$$\begin{array}{ccc} \mathrm{Mp}(\ell) \times \mathrm{Mp}(E) & \longrightarrow & \mathrm{Mp}(E) \\ \downarrow & & \downarrow \\ \mathrm{Sp}(\ell) \times \mathrm{Bp}(E) & \longrightarrow & \mathrm{Bp}(E). \end{array}$$

By restriction, we obtain the other

$$\begin{array}{ccc} \mathrm{Mp}_+(k, \ell) \times \mathrm{Mp}_+(I, E) & \longrightarrow & \mathrm{Mp}_+(I, E) \\ \downarrow & & \downarrow \\ \mathrm{Sp}_+(k, \ell) \times \mathrm{Bp}_+(I, E) & \longrightarrow & \mathrm{Bp}_+(I, E). \end{array}$$

Given the projection

$$\mathrm{Bp}_+(I, E) \longrightarrow \mathrm{Bp}_+(I, E)/Z = \mathrm{Bp}(N_I),$$

Lemma 6 now shows that $\mathrm{Mp}(N_I) =: \mathrm{Mp}_+(I, E)/Z$ is a metaplectic structure on N_I .

In the opposite direction, suppose given a metaplectic structure on N_I . Let $\mathcal{J}(E, \Omega_E)$ be the contractible space of all complex structures on E compatible with Ω_E [MS], and let us fix $J \in \mathcal{J}(E, \Omega_E)$. Thus E inherits a unitary structure, say $h = g + i\Omega$ where $g(v, w) = \Omega(v, Jw)$ and, by restriction of g , S is a Euclidean vector bundle. Let I^{\perp_g} and I^{\perp_h} be the Euclidean and Hermitian orthocomplements of I in E . The intersection $I^{\perp_h} = I^{\perp_g} \cap I^{\perp_\Omega}$ is a complex vector subbundle of E , linearly symplectomorphic to the vector bundle N_I . Having this identification in mind, let $\mathrm{BU}(N_I)$ be the bundle of all unitary frames in N_I . The metaplectic structure then yields an equivariant commutative diagram

$$\begin{array}{ccc} \mathrm{MU}(k) \times \widetilde{\mathrm{BU}}(N_I) & \longrightarrow & \widetilde{\mathrm{BU}}(N_I) \\ \downarrow & & \downarrow \\ \mathrm{U}(k) \times \mathrm{BU}(N_I) & \longrightarrow & \mathrm{BU}(N_I). \end{array}$$

Let $\mathrm{BO}_+(I) \subseteq \mathrm{BL}_+(I)$ be the subbundle of all oriented orthonormal frames of I . Let $\mathrm{BU}_+(I, E) = \mathrm{BU}(E) \cap \mathrm{BL}_+(I, E)$. We have an isomorphism of principal $\mathrm{U}_+(k, \ell)$ -bundles

$$\mathrm{Bp}(E) \supseteq \mathrm{BU}_+(I, E) \cong \mathrm{BO}_+(I) \times_N \mathrm{BU}(N_I).$$

Now $\widetilde{\text{BU}}_+(I, E) = \text{BO}_+(I) \times_N \widetilde{\text{BU}}(N_I)$ is an equivariant double cover of $\text{BU}_+(I, E)$ with respect to the double cover of structure groups

$$\text{SO}(k) \times \text{MU}(\ell - k) \rightarrow \text{U}_+(k, \ell),$$

and we obtain a metaplectic structure of E by group extension in view of Corollary 3.

It is clear that these two correspondences are one the inverse of the other.

Corollary 4. *Let E be a metaplectic vector bundle of rank 2ℓ . Let $S \subseteq E$ be an oriented rank- k isotropic subbundle. Then the structure group of E reduces to $\text{GL}_+(k) \times \text{Mp}(\ell - k) \subseteq \text{Mp}(\ell)$.*

If E is metaplectic vector bundle of rank 2ℓ , given the Segal-Shale-Weyl representation we may form the associated infinite dimensional vector bundle

$$\mathcal{S}(E) =: \widetilde{\text{Bp}}(E) \times_{\text{Mp}(\ell)} \mathcal{S}(\mathbb{R}^\ell).$$

If $I \subseteq E$ is an oriented rank- k isotropic subbundle, in view of the induced metaplectic structure on N_I we may similarly form infinite dimensional vector bundle

$$\mathcal{S}(N_I) =: \widetilde{\text{Bp}}(N_I) \times_{\text{Mp}(\ell - k)} \mathcal{S}(\mathbb{R}^{\ell - k}).$$

We have:

Corollary 5. *Let E be a metaplectic vector bundle of rank 2ℓ , and let $I \subseteq E$ be an oriented rank- k isotropic subbundle. Then the restriction map of Lemma 10 extends to a surjective morphism of vector bundles*

$$\Phi_I : \mathcal{S}(E) \longrightarrow \bigwedge^{-1/2} (I) \otimes_{\mathbb{C}} \mathcal{S}(N_I).$$

By the bilinear pairing (\cdot, \cdot) , we may view in the standard manner $\mathcal{S}(\mathbb{R}^\ell)$ as a subspace of the space $\mathcal{S}'(\mathbb{R}^\ell)$ of tempered distribution, which is invariant under the dual representation of $\text{Mp}(\ell)$ on $\mathcal{S}'(\mathbb{R}^\ell)$. If E is as above a rank- 2ℓ metaplectic vector bundle, let $\mathcal{S}_{\text{dual}}(E)$ denote the vector associated to the metaplectic structure and the dual action of $\text{Mp}(\ell)$ on $\mathcal{S}(\mathbb{R}^\ell)$. Since the metaplectic representation is unitary, we have an isomorphism

$$\mathcal{S}_{\text{dual}}(E) \cong \overline{\mathcal{S}(E)}.$$

Passing to conjugate bundles and inserting this isomorphism, we obtain from Corollary 5 a surjective morphism of vector bundles [BG]

$$\mathcal{S}_{\text{dual}}(E) \longrightarrow \overline{\bigwedge^{-1/2} (I) \otimes_{\mathbb{C}} \mathcal{S}_{\text{dual}}(N_I)}. \quad (11)$$

Consider now the inclusion of Lie groups

$$\mathrm{SO}(\ell) \subseteq \mathrm{GL}_+(\ell) \subseteq \mathrm{ML}(\ell) \subseteq \mathrm{Mp}(\ell).$$

The function $e^{-\|X\|^2} = e^{-\sum_i x_i^2} \in \mathcal{S}(\mathbb{R}^\ell)$ is a fixed point for the restriction of the Segal-Shale-Weyl representation to the subgroup $\mathrm{SO}(\ell)$. In the special case where E admits an oriented Lagrangian subbundle L , as we have mentioned the structure group reduces to $\mathrm{GL}_+(\ell)$. The choice of a compatible complex structure $J \in \mathcal{J}(E, \Omega_E)$ further reduces it to $\mathrm{SO}(\ell)$, the corresponding principal $\mathrm{SO}(\ell)$ -bundle $P_{L,J}$ being the bundle of all frames of the type

$$(e_1, \dots, e_\ell, J_p(e_1), \dots, J_p(e_\ell)),$$

where (e_1, \dots, e_ℓ) is an oriented orthonormal frame for $L(p)$, $p \in N$. Thus:

Lemma 11. *Suppose that the symplectic bundle (E, Ω_E) has an oriented Lagrangian subbundle $L \subseteq E$. Then to any $J \in \mathcal{J}(E, \Omega_E)$ there is associated a natural nowhere vanishing section $\sigma_{L,J}$ of $\mathcal{S}(E)$, which may be described as the constant $\mathcal{S}(\mathbb{R}^\ell)$ -valued function on $P_{L,J}$ equal to $e^{-\|x\|^2}$. Its image in $\Omega^{-1/2}(L)$ corresponds to the constant function 1 on $P_{L,J}$.*

In a related vein, the theory of [BG] also shows that the function $e^{-\|X\|^2}$ is a joint eigenvector for the metaplectic action of $\mathrm{MU}(\ell) =: s^{-1}(U(\ell)) \subseteq \mathrm{Mp}(\ell)$, i.e. there is a unitary character $c : \mathrm{MU}(\ell) \rightarrow \mathrm{U}(1)$ such that

$$v_{\mathrm{SSW}}(A) \left(e^{-\|X\|^2} \right) = c(A) \cdot e^{-\|X\|^2} \quad (A \in \mathrm{MU}(\ell).)$$

Thus, the function $e^{-\|X\|^2 - \|Y\|^2}$ is a fixed point for the tensor product action of $\mathrm{MU}(\ell)$ on $\mathcal{S}(\mathbb{R}^\ell) \otimes \mathcal{S}_{\mathrm{dual}}(\mathbb{R}^\ell) \cong \mathcal{S}(\mathbb{R}^\ell) \otimes \overline{\mathcal{S}(\mathbb{R}^\ell)}$. It represents the orthogonal projection of $\mathcal{S}(\mathbb{R}^\ell)$ onto the subspace $\mathrm{span} \left\{ e^{-\|X\|^2} \right\}$.

Now the choice of a compatible complex structure $J \in \mathcal{J}(E, \Omega_E)$ reduces the structure group of E to $U(\ell)$; let $\mathrm{Bu}(E, J) \subseteq \mathrm{Bl}(E)$ be the principal $U(\ell)$ -bundle of all unitary frames in E (for (Ω_E, J)). If (E, Ω_E) in addition is metaplectic, with metaplectic structure $\widetilde{\mathrm{Bp}}(E) \xrightarrow{\varpi} \mathrm{Bp}(E)$, the inverse image $\widetilde{\mathrm{Bu}}(E, J) =: \varpi^{-1}(\mathrm{Bu}(E, J))$ is a principal $\mathrm{MU}(\ell)$ -bundle. Thus, we may globalize the previous construction to obtain:

Lemma 12. *Let (E, Ω_E) be a metaplectic vector bundle. Then to each $J \in \mathcal{J}(E, \Omega_E)$ there is associated a line subbundle $L_J \subseteq \mathcal{S}(E)$, which in the trivialization of $\mathcal{S}(E)$ offered by any element of $\widetilde{\mathrm{Bu}}(E, J)$ is the line spanned by $e^{-\|X\|^2}$. In the same trivialization, the orthogonal projector $\mathcal{S}(E) \rightarrow L_J$, as a section of the bundle of linear endomorphisms of $\mathcal{S}(E)$, is represented by the function $e^{-\|X\|^2 - \|Y\|^2}$.*

When operating on symbols of Szegő kernels and related distributions, a naturally occurring case is that of symplectic vector bundles of the form $E^+ \oplus E^-$, as in the following Lemma (that we simply state):

Lemma 13. *i) Let $E^+ = (E, \Omega_E)$ and $F^+ = (F, \Omega_F)$ be metaplectic vector bundles on a manifold N . Then there is a naturally induced metaplectic structure on their direct sum $E^+ \oplus F^+ = (E \oplus F, \Omega_E \oplus \Omega_F)$, and a natural isomorphism*

$$\mathcal{S}(E^+ \oplus F^+) \cong \mathcal{S}(E) \otimes \mathcal{S}(F).$$

ii) Let $E^+ = (E, \Omega_E)$ be a metaplectic vector bundle. Then there is a naturally induced metaplectic structure on its opposite $E^- =: (E, -\Omega_E)$, and a natural isomorphism

$$\mathcal{S}(E^-) \cong \mathcal{S}_{\text{dual}}(E^+).$$

iii) In particular, if $E^+ = (E, \Omega_E)$ is a metaplectic vector bundle then there is a naturally induced metaplectic structure on $E^+ \oplus E^-$, and a natural isomorphism

$$\mathcal{S}(E^+ \oplus E^-) \cong \text{End}_{\mathcal{HS}}(\mathcal{S}(E)),$$

where the latter denotes the vector bundle of Hilbert-Schmidt linear endomorphisms of $\mathcal{S}(E)$.

If $S = E^+ \oplus E^-$, for any $J \in \mathcal{J}(E, \Omega_E)$ we have

$$J_S =: J \oplus (-J) \in \mathcal{J}(S, \Omega_E \oplus (-\Omega_E)).$$

Any unitary frame $(e_1, \dots, e_\ell) \in \text{Bu}(E, J) = \text{Bu}(E, -J)$ extends to a unitary frame $(\underbrace{e_1, \dots, e_\ell}_{\widetilde{e_1, \dots, e_\ell}}, \underbrace{e_1, \dots, e_\ell}_{\widetilde{e_1, \dots, e_\ell}}) \in \text{Bu}(N, J_N)$. Passing to metaplectic double covers, $\text{Bu}(E, J) \subseteq \widetilde{\text{Bu}}(S, J_S)$. Summing up, we have:

Corollary 6. *Suppose that the symplectic vector bundle (E, Ω_E) is metaplectic. Set $S =: E^+ \oplus E^-$. Then $\mathcal{S}(S) \cong \text{End}_{\mathcal{HS}}(\mathcal{S}(E))$ has a distinguished section σ_J for every compatible complex structure $J \in \mathcal{J}(E, \Omega_E)$. In any appropriate trivialization (in the sense above), this is represented by the function $e^{-\|X\|^2 - \|Y\|^2} \in \mathcal{S}(\mathbb{R}^{2\ell})$.*

We now look at the image of the distinguished section in Corollary 6 under the morphism of vector bundles in Corollary 5. The proof of the following is a case by case application of the local form in Lemma 10 of the vector bundle morphism of Corollary 5, and is left to the reader:

Corollary 7. *In the hypothesis of Corollary 6, suppose that $L \subseteq E$ is an oriented isotropic subbundle, and consider the isotropic subbundle $\text{diag}(L) = \{(l, l) : l \in L\} \subseteq S$. Fix $J \in \mathcal{J}(E, \Omega_E)$, and let h be the Hermitian structure on E associated to the compatible pair (Ω_E, J) . Let $g = \Re(h)$ be the associated Riemannian metric. Let $L^\perp \subseteq E$ be the symplectic annihilator of L in (E, Ω_E) , and let L^0 be the Riemannian orthocomplement of L in (E, g) . Then:*

- i): *The intersection $L^\perp \cap L^0$ is a complex vector subbundle of (E, Ω_E) , whence a symplectic subbundle of (E, Ω_E) ;*
- ii): *The symplectic normal bundle $N_L = L^\perp / L$ of L in E^+ is (naturally) symplectically isomorphic to $L^\perp \cap L^0$ (we shall henceforth not distinguish between N_L and $L^\perp \cap L^0$), and thus has a naturally induced compatible complex structure J_{N_L} ;*
- iii): *Let us endow $S = E^+ \oplus E^-$ with the compatible complex structure $J \oplus (-J)$. Then the symplectic normal bundle $N_{\text{diag}(L)}$ of $\text{diag}(L)$ in S is (naturally isomorphic to) the direct sum of vector subbundles*

$$\begin{aligned}
N_{\text{diag}(L)} &\cong (\{(l, -l) : l \in L\} \oplus \{(Jl, Jl) : l \in L\}) \\
&\quad \oplus (N_L^+ \oplus N_L^-) \\
&= (L_r \oplus L_i) \oplus (N_L^+ \oplus N_L^-) \\
&= L_{\mathbb{C}} \oplus (N_L^+ \oplus N_L^-).
\end{aligned} \tag{12}$$

Here L_r and L_i , defined by the second identity, are oriented Lagrangian subbundles of the complex vector subbundle $L_{\mathbb{C}} =: L_r \oplus L_i \subseteq S$. Since $L_r \cong L$ is oriented, $L_{\mathbb{C}}$ is metaplectic.

- iv): *Let $\Phi_{\text{diag}(L)} : \mathcal{S}(S) \rightarrow \bigwedge^{-1/2}(L) \otimes \mathcal{S}(N_{\text{diag}(L)})$ be the vector bundle morphism introduced in Corollary 5. Let σ_J be the distinguished section of $\mathcal{S}(S)$ associated to the complex structure J as in Corollary 6. Then*

$$\Phi_{\text{diag}(L)}(\sigma_J) = \text{vol}_L^{-1/2} \otimes \sigma_{L_r} \otimes \sigma_{J_{N_L}}.$$

Here $\text{vol}_L^{-1/2}$ is the section of $\bigwedge^{-1/2}(I)$ taking value one on oriented orthonormal basis of I , σ_{L_r} is the section of $\mathcal{S}(L_{\mathbb{C}})$ associated to the Lagrangian subbundle L_r according to Lemma 11, and $\sigma_{J_{N_L}}$ is the section of $\mathcal{S}(N_L^+ \oplus N_L^-)$ associated to the complex structure J_{N_L} on N_L , according to Corollary 6.

- v): *The symplectic normal bundle of L_r in $N_{\text{diag}(L)}$ is isomorphic as a unitary vector bundle to $N_L^+ \oplus N_L^-$. Let $\Phi_{L_r} : \mathcal{S}(N_{\text{diag}(L)}) \rightarrow \bigwedge^{-1/2}(L_r) \otimes \mathcal{S}(N_{L_r})$*

be the vector bundle morphism from Corollary 5. Then

$$\Phi_{L_r} \left(\sigma_{L_r} \otimes \sigma_{J_{N_L}} \right) = \text{vol}_{L_r}^{-1/2} \otimes \sigma_{J_{N_L}}.$$

3 The geometry of the symbol calculus

We need to recall some basic constructions from [BG], at places rephrasing them in terms of the principal bundles involved in our constructions. Let A and B be \mathcal{C}^∞ manifolds. There is a natural symplectomorphism $T^*(A \times B) \cong T^*(A) \times T^*(B)$ (cotangent bundles are implicitly endowed with their canonical symplectic structures), which will be implicit throughout. Suppose that $\Gamma' \subseteq T^*(A \times B) \setminus \{0\}$ is a closed Lagrangian conic submanifold. Let $\Gamma \subseteq T^*(A) \times T^*(B) \setminus \{0\}$ be the associated canonical relation, defined as the image of Γ' under the involution $((a, \eta), (b, \vartheta)) \mapsto ((a, \eta), (b, -\vartheta))$. Suppose that $\Sigma \subseteq T^*(B) \setminus \{0\}$ is a closed isotropic conic submanifold. Let us form the fibre diagram

$$\begin{array}{ccc} F & \xrightarrow{\rho} & \Gamma \\ \varrho \downarrow & & \downarrow \gamma \\ \Sigma & \xrightarrow{\iota} & T^*(B) \setminus \{0\}. \end{array}$$

Here ι is the inclusion, and γ the projection. Thus,

$$F = \{(\sigma, (\tau, \sigma)) : \sigma \in \Sigma, (\tau, \sigma) \in \Gamma\}.$$

We have a diffeomorphism

$$F \cong q_\Gamma^{-1}(\Sigma) = q^{-1}(\Sigma) \cap \Gamma, \quad (\sigma, (\tau, \sigma)) \leftrightarrow (\tau, \sigma),$$

so that F maps naturally into Γ , diffeomorphically onto its image. We shall implicitly think of F as embedded in Γ , as the subset of all pairs $(\tau, \sigma) \in \Gamma$ with $\sigma \in \Sigma$.

Let us assume that all the clean intersection and properness hypothesis in Chapter 7 of *loc. cit.* are satisfied (this will be always the case in our situation). Given the projections

$$\begin{array}{ccc} \Gamma & & T^*(A \times B) \\ p_\Gamma \swarrow & & \swarrow p \\ T^*(A) & & T^*(A) \\ q_\Gamma \searrow & & \searrow q \\ & & T^*(B) \end{array}$$

we have a conic isotropic submanifold

$$\begin{aligned}\Gamma \circ \Sigma &= \{\tau \in T^*(A) \setminus \{0\} : \exists \sigma \in \Sigma \text{ such that } (\tau, \sigma) \in \Gamma\} \\ &= p(q^{-1}(\Sigma) \cap \Gamma) \\ &= p_\Gamma(q_\Gamma^{-1}(\Sigma)) \subseteq T^*(A) \setminus \{0\}.\end{aligned}$$

Furthermore, the projection $p_F : F \rightarrow \Gamma \circ \Sigma$, $(\sigma, (\tau, \sigma)) \mapsto \tau$, is a fibration with compact fibres.

Let us define a vector bundle U_0 on F by

$$U_0(\tau, \sigma) = \{w \in T_\sigma \Sigma : (0, w) \in T_{(\tau, \sigma)} \Gamma\}.$$

Since p_F is a submersion, if $w \in U_0(\tau, \sigma)$ then $(0, w) \in T_{(\tau, \sigma)} F$ is tangent to the fibre of p_F over τ . In other words, $TF \supseteq U_0 = \ker(dp_F)$ is the vertical tangent bundle of p_F .

The functorial behaviour of the symbol of Fourier-Hermite generalized half-forms is governed by a *symbol map*, which transforms symplectic spinors on Σ into symplectic spinors on $\Gamma \circ \Sigma$, and whose existence is the content of Proposition 6.5 of [BG]. More precisely, in the present context this is a surjective morphism of vector bundles on F ,

$$\Psi_{\Sigma, \Gamma} : \rho^* \left(\bigwedge^{1/2}(\Gamma) \right) \otimes_{\mathbb{C}} \varrho^*(\text{Spin}(\Sigma)) \longrightarrow \det(U_0^*) \otimes p_F^*(\text{Spin}(\Gamma \circ \Sigma)).$$

In the sequel, we shall need to reformulate its construction, in order to have an explicit description of the symbol map in terms of the principal bundles involved.

Since in our applications A, B, Γ, Σ, F , the vector bundle U_0 above and the vector bundle U introduced below will all be orientable, we shall make this simplifying assumption throughout. It will also simplify our exposition to assume, as will be the case in our applications, that $p_F : F \rightarrow \Gamma \circ \Sigma$ is a Riemannian submersion, having therefore a natural connection, and that TF has a natural (oriented) complement in $T\Gamma|_F$, denoted $N_F|_\Gamma$.

Given that B is orientable, hence metalear, T^*B is metaplectic. Since $\Sigma \subseteq T^*B \setminus \{0\}$ is an oriented isotropic submanifold, its symplectic normal bundle in T^*B , $N_\Sigma = (T\Sigma)^\perp / T\Sigma$, has an induced metaplectic structure. The spinor bundle of Σ is

$$\text{Spin}(\Sigma) = \bigwedge^{1/2}(\Sigma) \otimes \mathcal{S}(N_\Sigma).$$

The same considerations apply to the conic isotropic submanifold $\Gamma \circ \Sigma \subseteq T^*A \setminus \{0\}$, with associated spinor bundle

$$\text{Spin}(\Gamma \circ \Sigma) = \bigwedge^{1/2} (\Gamma \circ \Sigma) \otimes \mathcal{S}(N_{\Gamma \circ \Sigma}).$$

Let us define a second vector bundle $U_1 \supseteq U_0$ on F by

$$U_1(\tau, \sigma) = \{w \in (T_\sigma \Sigma)^\perp : (0, w) \in T_{(\tau, \sigma)} \Gamma\},$$

for $(\tau, \sigma) \in F$.

Then U_0 and U_1 are isotropic vector subbundles of the symplectic vector bundle $\varrho^*(T(T^*B))$, and their quotient $U = U_1/U_0 \subseteq \varrho^*(N_\Sigma)$ is an isotropic vector subbundle, which as mentioned we shall assume oriented. By Corollary 5, therefore, the structure group of the metaplectic bundle of N_Σ , $\widetilde{\text{Bp}}(N_\Sigma)$, reduces to

$$\text{Bl}_+(U) \times \widetilde{\text{Bp}}(U^\perp/U),$$

and there is furthermore a surjective morphism of vector bundles associated to this reduced principal bundle,

$$G_{\Sigma, U} : \mathcal{S}(N_\Sigma) \longrightarrow \bigwedge^{-1/2} (U) \otimes \mathcal{S}(U^\perp/U).$$

By construction, $G_{\Sigma, U}$ is in the appropriate trivialization the restriction map of Lemma 10.

By Proposition 6.4 of [BG], $N_{\Gamma \circ \Sigma} \cong U^\perp/U$ naturally, so that there is at any rate a morphism of vector bundles on F from $\rho^*\left(\bigwedge^{1/2}(\Gamma)\right) \otimes_{\mathbb{C}} \varrho^*(\text{Spin}(\Sigma))$ to

$$\rho^*\left(\bigwedge^{1/2}(\Gamma)\right) \otimes \varrho^*\left(\bigwedge^{1/2}(\Sigma)\right) \otimes \bigwedge^{-1/2}(U) \otimes p_F^*(\mathcal{S}(N_{\Gamma \circ \Sigma})). \quad (13)$$

We shall therefore be done by exhibiting an isomorphism of line bundles

$$\rho^*\left(\bigwedge^{1/2}(\Gamma)\right) \otimes \varrho^*\left(\bigwedge^{1/2}(\Sigma)\right) \otimes \bigwedge^{-1/2}(U) \rightarrow \det(U_0^*) \otimes p_F^*\left(\bigwedge^{1/2}(\Gamma \circ \Sigma)\right). \quad (14)$$

For the rest of this proof, we shall fix a point $f = (\tau, \sigma) \in F$, and write for notational simplicity $T\Sigma$, $T\Gamma$, and so forth for $T_f F$, $T_f \Gamma$, and so forth. Similarly, U_0 , U_1 , U will be the fibres at f of the corresponding vector bundles

on F , and the same provision will apply to the various principal bundles involved.

Given the assumptions discussed above, we have direct sum decompositions

$$T\Gamma \cong TF \oplus N \cong U_0 \oplus T(\Gamma \circ \Sigma) \oplus N,$$

whence we may reduce the principal bundle of the line bundles on both sides of (14) to the product

$$\mathrm{Bl}_+(U_0) \times \mathrm{Bl}_+(T(\Gamma \circ \Sigma)) \times \mathrm{Bl}_+(N) \times \mathrm{Bl}_+(\Sigma) \times \mathrm{Bl}_+(U),$$

whose general element we shall denote by $(e_{U_0}, e_{\Gamma \circ \Sigma}, e_N, e_\Sigma, e_U)$.

Let U_1^\perp be the symplectic annihilator of U_1 in $T(T^*B)$. Let $\gamma : T\Gamma \oplus T\Sigma \rightarrow U_1^\perp$ be given by $\gamma((v, w), w') = w - w'$. We then have a short exact sequence ([BG], page 45):

$$0 \longrightarrow TF \longrightarrow T\Gamma \oplus T\Sigma \xrightarrow{\gamma} U_1^\perp \longrightarrow 0.$$

Under γ , the subspace $N \oplus T\Sigma \subseteq T\Gamma \oplus T\Sigma$ maps isomorphically onto $U_1^\perp \cong (T(T^*B)/U_1)^*$. Given a pair $(e_N, e_\Sigma) \in \mathrm{Bl}_+(N) \times \mathrm{Bl}_+(\Sigma)$, we shall denote by $f_{(e_N, e_\Sigma)}$ the oriented basis of $T(T^*B)/U_1$ dual to the oriented basis $\gamma((e_N, e_\Sigma))$ of $(T(T^*B)/U_1)^*$.

Let $\Omega_{T^*B, \mathrm{can}}$ denote the canonical symplectic structure of T^*B , $\mathrm{vol}_{T^*B} = \Omega_{T^*B, \mathrm{can}}^b$ the associated volume form, and $\mathrm{vol}_{T^*B}^{1/2} = \sqrt{\Omega_{T^*B, \mathrm{can}}^b}$ the associated half-form. If now we are given $\vartheta_\Gamma \in \Lambda^{1/2}(\Gamma)$, $\vartheta_\Sigma \in \Lambda^{1/2}(\Sigma)$, $\vartheta_U^{-1} \in \Lambda^{-1/2}(U)$, the expression

$$\vartheta_\Gamma(e_{U_0}, e_{\Gamma \circ \Sigma}, e_N) \cdot \vartheta_\Sigma(e_\Sigma) \cdot \vartheta_U^{-1}(e_U) \cdot \mathrm{vol}_{T^*B}^{1/2}(e_{U_0}, e_U, f_{(e_N, e_\Sigma)}) \quad (15)$$

only depends on $(e_{U_0}, e_{\Gamma \circ \Sigma}) \in \mathrm{Bl}_+(U_0) \times \mathrm{Bl}_+(\Gamma \circ \Sigma)$.

We thus have:

Proposition 2. *Assume that U_0 is oriented. Then the previous construction defines a surjective morphism of vector bundles on F*

$$\Psi_{\Sigma, \Gamma} : \rho^* \left(\bigwedge^{1/2}(\Gamma) \right) \otimes_{\mathbb{C}} \varrho^*(\mathrm{Spin}(\Sigma)) \longrightarrow \det(U_0^*) \otimes p_F^*(\mathrm{Spin}(\Gamma \circ \Sigma)).$$

Our additional hypothesis that U_0 be orientable accounts for the appearance of the determinant line bundle in place of the line bundle of densities.

4 Restricting and pushing forward $\tilde{\Pi}_X$.

Let L^* be the dual line bundle to L , endowed with the induced Hermitian metric. Let $X \subseteq L^*$ be the unit circle bundle, $H(X) \subseteq \mathcal{C}^\infty(X)$ be the Hardy space of boundary values of holomorphic functions. The S^1 -action induces a decomposition $H(X) = \bigoplus_{k \in \mathbb{N}} H_k(X)$, and the k -th isotype $H_k(X)$ is canonically isomorphic to $H^0(M, L^{\otimes k})$. Given that Ω is symplectic, the connection form α is a contact structure on X . Given that Hermitian metric h on L is G -invariant, the action of G on L^* leaves X invariant.

The Kähler manifold (M, Ω) and the contact manifold (X, α) have natural volume forms $\text{vol}_M = \Omega^{\wedge n}$ and $\text{vol}_X = \pi_X^*(\text{vol}_M) \wedge \alpha$, respectively. Let $\text{vol}_M^{(1/2)} = \sqrt{\text{vol}_M}$ and $\text{vol}_X^{(1/2)} = \sqrt{\text{vol}_X}$ denote the respective associated half-forms. The volume form vol_X makes the space of smooth functions on X into a prehilbert vector space, unitarily isomorphic to the space $\Omega^{1/2}(X)$ of smooth half-forms under the map $f \mapsto f \text{vol}_X^{(1/2)}$. This isomorphism will be implicit throughout, and we shall accordingly view the Szegő kernel $\tilde{\Pi}_X$, rather than as a generalized density, as a generalized half-form on $X \times X$: $\tilde{\Pi}_X \in \mathcal{D}'_{1/2}(X \times X)$. Similarly, taking products we obtain volume forms $\text{vol}_{M \times M}$ and $\text{vol}_{X \times X}$ on $M \times M$ and $X \times X$, respectively, with associated half-forms $\text{vol}_{M \times M}^{(1/2)}$ and $\text{vol}_{X \times X}^{(1/2)}$.

In general, if f_1 and f_2 are functions, or sections, or half-forms, and so forth, on manifolds A_1 and A_2 , we shall denote by $f_1 \boxtimes f_2 = \pi_1^*(f_1) \otimes \pi_2^*(f_2)$ the corresponding object on the product $A_1 \times A_2$ obtained by pull-back under the projections $\pi_i : A_1 \times A_2 \rightarrow A_i$ and tensor product. Thus, we may write the Szegő kernel as

$$\tilde{\Pi}_X = \sum_{k,j} s_j^{(k)} \boxtimes \overline{s_j^{(k)}} \cdot \text{vol}_X^{(1/2)} \boxtimes \text{vol}_X^{(1/2)},$$

where for every $k = 1, 2, \dots$ $\{s_j^{(k)}\}$ ($1 \leq j \leq h^0(M, L^{\otimes k})$) is an orthonormal basis for the k -th Fourier component $H(X)_k \cong H^0(M, L^{\otimes k})$ of the Hardy space $H(X)$.

Let $M' =: \Phi^{-1}(0) \subseteq M$. If $\pi_{X|M} : X \rightarrow M$ is the projection, let $X' =: \pi_{X|M}^{-1}(M')$ and denote by $j_{X'} : X' \hookrightarrow X$ be the inclusion. Under the above assumptions, $p_{X'} : X' \rightarrow X_0 =: X'/G$ is a principal G -bundle. Clearly, X_0 is the unit circle bundle in L_0^* with the induced metric h_0 . We shall now endow the inclusion $j_{X'}$ and the projection $p_{X'}$ with the structure of morphisms of metilinear manifolds; by taking products, this will also make the inclusion $j_{X' \times X'} : X' \times X' \rightarrow X \times X$ and the projection $\pi_{X'} : X' \times X' \rightarrow X_0 \times X_0$ into morphisms of metilinear manifolds. We shall then apply the associated pull-back and push-forward operations to the Szegő kernel $\tilde{\Pi}_X \in \mathcal{D}'_{1/2}(X \times X)$.

By using the microlocal theory of [BG], we shall relate the result to the Szegő kernel of the symplectic quotient X_0 , $\Pi_{X_0} \in \mathcal{D}'_{1/2}(X_0 \times X_0)$.

To this end, let us fix an orientation on G , and thus on its Lie algebra, \mathfrak{g} . In particular, this makes M' and X' into oriented Riemannian manifolds. As discussed in section 2, there is then a natural way to restrict a half-form on M (or X) to a half-form on M' (respectively, on X'), namely by setting $\iota^*(\eta_M) = \eta_{M'}$ (respectively, $j^*(\eta_X) = \eta_{X'}$). This makes ι and j into morphisms of metalelinear manifolds, $\tilde{\iota}$ and \tilde{j} .

Taking products, we clearly obtain restriction maps $\iota^* \otimes \iota^*$ and $j^* \otimes j^*$ on half-form bundles and $(\iota \times \iota)^* : \Omega^{1/2}(M \times M) \rightarrow \Omega^{1/2}(M' \times M')$, $(j \times j)^* : \Omega^{1/2}(X \times X) \rightarrow \Omega^{1/2}(X' \times X')$ on smooth global half-forms.

Let $\Gamma_{j \times j} \subseteq T^*(X \times X) \times T^*(X' \times X')$ be the canonical relation associated to $j \times j$. The morphism of metalelinear manifolds $\tilde{j} \times \tilde{j}$ is equivalent to the assignment of the half-form $\text{vol}_{\Gamma_{j \times j}}^{1/2}$ described in (5), with $f = j \times j$, $P = X' \times X'$ and $Q = X \times X$.

By the same token, the principal G -bundle $p_{X'} : X' \rightarrow X_0 = X'/G$ is also a morphism of metalelinear manifolds in the natural manner: $p_{X'}^*(\eta_{X_0}) = \eta_{X'}$. Let $p_{X'*} : \Omega^{1/2}(X') \rightarrow \Omega^{1/2}(X_0)$ denote the resulting push-forward operator. For any smooth function f on X' , let us denote by f^G the G -invariant component of f , when decomposed over the irreducible representations of G . Thus f^G may be viewed implicitly as a function on $X_0 = X'/G$. Furthermore, let us write V_{eff} for the *effective potential* of the action, that is, $V_{\text{eff}}(\bar{x})$ is the volume of the fibre $p_{X'}^{-1}(\bar{x})$, $\bar{x} \in X_0$ [BG]. By Lemma 1, if $\bar{x} \in X_0$ then

$$\begin{aligned} p_{X'*} \left(f \text{vol}_{V(X'/X_0)}^{(1/2)} \otimes \text{vol}_{H(X'/X_0)}^{(1/2)} \right) (\bar{x}) \\ = \left(\int_{p_{X'}^{-1}(\bar{x})} f \cdot \text{vol}_{V(X'/X_0)}^{(1/2)} \cdot \overline{\text{vol}_{V(X'/X_0)}^{(1/2)}} \right) \cdot \text{vol}_{X_0}^{(1/2)} \\ = f^G(\bar{x}) \cdot V_{\text{eff}}(\bar{x}) \cdot \text{vol}_{X_0}^{(1/2)} \end{aligned}$$

Taking products, the principal $G \times G$ -bundle $p_{X' \times X'} = p_{X'} \times p_{X'} : X' \times X' \rightarrow X_0 \times X_0$ can be similarly made into a morphism of metalelinear manifolds, with an attached push-forward operation $p_{X' \times X'} : \Omega^{1/2}(X' \times X') \rightarrow \Omega^{1/2}(X \times X)$. If f, g are smooth functions on X' and $(f \boxtimes g)(x, y) =: f(x) \cdot g(y)$ ($x, y \in X'$), then $(f \boxtimes g)^{G \times G} = f^G \boxtimes g^G$. Furthermore, the effective potential for the product is $V'_{\text{eff}} = V_{\text{eff}} \boxtimes V_{\text{eff}}$, that is, $V'_{\text{eff}}(\bar{x}, \bar{y}) = V_{\text{eff}}(\bar{x}) \cdot V_{\text{eff}}(\bar{y})$ ($\bar{x}, \bar{y} \in X_0$).

Thus,

Lemma 14. $\text{vol}_{X' \times X'}^{(1/2)} = \text{vol}_{X'}^{(1/2)} \boxtimes \text{vol}_{X'}^{(1/2)}$, and

$$p_{X' \times X'*} \left(f \boxtimes g \text{vol}_{X' \times X'}^{(1/2)} \right) = f^G \boxtimes g^G \cdot V_{\text{eff}} \boxtimes V_{\text{eff}} \cdot \text{vol}_{X_0}^{(1/2)} \boxtimes \text{vol}_{X_0}^{(1/2)},$$

for any pair of smooth functions f and g on X_0 .

Proof. This follows from the equalities

$$\begin{aligned} p_{X' \times X'}^* \left(f \boxtimes g \operatorname{vol}_{X' \times X'}^{(1/2)} \right) &= p_{X' \times X'}^* \left((f \cdot \operatorname{vol}_{X'}^{(1/2)}) \boxtimes (g \operatorname{vol}_{X'}^{(1/2)}) \right) \\ &= (f^G V_{\text{eff}} \operatorname{vol}_{X_0}^{(1/2)}) \boxtimes (g^G V_{\text{eff}} \cdot \operatorname{vol}_{X_0}^{(1/2)}). \end{aligned}$$

Since $p_{X' \times X'}$ is proper, the push-forward $p_{X' \times X'}^*$ extends to a continuous linear map of Fréchet vector spaces

$$p_{X' \times X'}^* : \mathcal{D}'_{1/2}(X' \times X') \longrightarrow \mathcal{D}'_{1/2}(X_0 \times X_0).$$

Let furthermore $\Sigma \subseteq T^*(X \times X) \setminus \{0\}$ be the wave front set of the Szegő kernel, that is,

$$\Sigma = \{(x, x, r\alpha_x, -r\alpha_x) : x \in X, r > 0\}.$$

Let $\mathcal{D}'_{1/2}(X \times X)_\Sigma \subseteq \mathcal{D}'_{1/2}(X \times X)$ be the subspace of all generalized half-forms on $X \times X$ having wave front contained in Σ , with the appropriate topology. Let

$$N_{j \times j} = \{(x_1, x_2, \eta_1, \eta_2) : x_i \in X', \eta_i = 0 \text{ on } T_{x_i} X' \subseteq T_{x_i} X\}$$

be the *conormal bundle* of the embedding $j \times j$. Since $\Sigma \cap N_{j \times j} = \emptyset$, the pull-back $(j \times j)^*$ extends to a continuous linear map of Fréchet vector spaces ([D], Proposition 1.3.3)

$$(j \times j)^* : \mathcal{D}'_{1/2}(X \times X)_\Sigma \longrightarrow \mathcal{D}'_{1/2}(X' \times X').$$

By composition, we thus have a well-defined continuous linear map

$$\Upsilon =: p_{X' \times X'}^* \circ (j \times j)^* : \mathcal{D}'_{1/2}(X \times X)_\Sigma \longrightarrow \mathcal{D}'_{1/2}(X_0 \times X_0).$$

Let us define, for $k \in \mathbb{Z}$,

$$\begin{aligned} \mathcal{D}'_{1/2}(X_0 \times X_0)_k &= \{\varphi \in \mathcal{D}'(X_0 \times X_0) : \\ &\quad u(e^{i\theta} x, y) = e^{ik\theta} u(x, y) \forall e^{i\theta} \in S^1, \forall x, y \in X\}, \end{aligned}$$

and similarly on $\mathcal{D}'_{1/2}(X \times X)$ and $\mathcal{D}'_{1/2}(X' \times X')$. Let $T_k : \mathcal{D}'_{1/2}(X_0 \times X_0) \longrightarrow \mathcal{D}'_{1/2}(X_0 \times X_0)_k$ be given by

$$T_k(u) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} u(e^{i\theta} x, y) d\theta.$$

Then

$$\mathcal{D}'_{1/2}(X_0 \times X_0) \cong \bigoplus_{k=-\infty}^{+\infty} \mathcal{D}'_{1/2}(X_0 \times X_0)_k,$$

and T_k represents the projection onto the k -th factor.

We now focus on the generalized half-form

$$\tilde{\Pi}_{X|X_0} =: \Upsilon \left(\tilde{\Pi}_X \right) \in \mathcal{D}'_{1/2}(X_0 \times X_0).$$

Since j and $p_{X'}$ are S^1 -equivariant maps, we have for every $k \in \mathbb{Z}$:

$$T_k \left(\Upsilon \left(\tilde{\Pi}_X \right) \right) = \Upsilon \left(T_k(\tilde{\Pi}_X) \right).$$

Clearly, $T_k(\tilde{\Pi}_X)$ is the distributional kernel of the orthogonal projector onto the subspace $H(X)_k \cong H^0(M, L^{\otimes k})$.

For every $k = 0, 1, 2, \dots$ let $\left\{ s_j^{(G,k)} \right\}_{1 \leq j \leq d_k}$ denote an orthonormal basis of the space of G -invariant Hardy functions of level k , $H(X)_k^G \cong H^0(M, L^{\otimes k})^G$. By Lemma 14, we have

$$\tilde{\Pi}_{X|X_0}(x_0, y_0) = V_{\text{eff}}(x_0)V_{\text{eff}}(y_0) \sum_{k=0}^{+\infty} \sum_{j=1}^{d_k} s_j^{(G,k)}(x') \otimes \overline{s_j^{(G,k)}(y')}, \quad (16)$$

if $x_0, y_0 \in X_0$ and $x', y' \in X'$ lie over x_0, y_0 respectively.

We shall now use the theory of [BG] to describe $\tilde{\Pi}_{X|X_0}$ as a Fourier-Hermite generalized half-form. Namely, let us first recall that, as a Fourier-Hermite generalized half-form, the Szegő kernel satisfies $\tilde{\Pi}_X \in J^{1/2}(X \times X, \Sigma)$. Let furthermore α_0 be the connection 1-form on X_0 , and set

$$\Sigma_0 = \{(x, x, r\alpha_{0x}, -r\alpha_{0x}) : x \in X_0, r > 0\} \subseteq T^*(X_0 \times X_0) \setminus \{0\}.$$

Lemma 15. $\tilde{\Pi}_{X|X_0} \in J^{(1+g)/2}(X_0 \times X_0, \Sigma_0)$.

Proof. Referring to chapter 7 of [BG], let us consider the fibre diagram associated to $\Sigma \subseteq T^*(X \times X) \setminus \{0\}$ and the conormal bundle $\Gamma'_{j \times j} \subseteq T^*(X' \times X' \times X \times X) \setminus \{0\}$ of the graph of $j \times j$:

$$\begin{array}{ccc} F_{j \times j} & \longrightarrow & \Gamma_{j \times j} \\ \downarrow & & \downarrow \\ \Sigma & \longrightarrow & T^*(X \times X), \end{array} \quad (17)$$

where $\Gamma_{j \times j}$ denotes the image of $\Gamma'_{j \times j}$ under sign reversal in the first component of $T^*((X' \times X') \times (X \times X))$. We have

$$\Gamma_{j \times j} = \left\{ (x_1, x_2, j(x_1), j(x_2), (d_{x_1}j)^t(\eta_1), (d_{x_2}j)^t(\eta_2), \eta_1, \eta_2) : \right. \\ \left. x_i \in X', \eta_i \in T^*_{j(x_i)}(X), i = 1, 2. \right\}. \quad (18)$$

Let us set $\alpha' =: j^*(\alpha)$. Then

$$F_{j \times j} = \left\{ (x, x, j(x), j(x), r\alpha'_x, -r\alpha'_x, r\alpha_{j(x)}, -r\alpha_{j(x)}) : x \in X', r > 0. \right\}.$$

Thus $F_{j \times j}$ is diffeomorphic to its projection

$$\Sigma' = \Gamma_{j \times j} \circ \Sigma = \left\{ (x, x, r\alpha'_x, -r\alpha'_x) : x \in X', r > 0. \right\} \subseteq T^*(X' \times X') \setminus \{0\}.$$

Therefore, the excess of the diagram (17) is

$$\begin{aligned} e_{j \times j} &= \dim(F_{j \times j}) + \dim T^*(X \times X) - \dim(\Gamma'_{j \times j}) - \dim(\Sigma) \\ &= g. \end{aligned}$$

Hence, given that all the clean intersection hypothesis of Theorem 9.1 of [BG] are satisfied, we have $(j \times j)^*(\tilde{\Pi}_X) \in J^{(1+g)/2}(X' \times X', \Sigma')$.

Let us next consider the fibre diagram associated to $p_{X'} \times p_{X'}$. The canonical relation is now

$$\Gamma_{j \times j} = \left\{ (x_1, x_2, p_{X'}(x_1), p_{X'}(x_2), (d_{x_1}p_{X'})^t(\eta_1), (d_{x_2}p_{X'})^t(\eta_2), \eta_1, \eta_2) : \right. \\ \left. x_i \in X', \eta_i \in T^*_{p_{X'}(x_i)}(X_0), i = 1, 2. \right\}. \quad (19)$$

The fibre diagram is

$$\begin{array}{ccc} F_{p_{X'} \times p_{X'}} & \longrightarrow & \Gamma_{p_{X'} \times p_{X'}} \\ \downarrow & & \downarrow \\ \Sigma' & \longrightarrow & T^*(X' \times X'). \end{array} \quad (20)$$

Here $F_{p_{X'} \times p_{X'}}$ is clearly diffeomorphic to Σ' . Thus, the excess is now

$$\begin{aligned} e_{p_{X'} \times p_{X'}} &= \dim(F_{p_{X'} \times p_{X'}}) + \dim T^*(X' \times X') - \dim(\Gamma_{p_{X'} \times p_{X'}}) - \dim(\Sigma') \\ &= 2g. \end{aligned}$$

Given that the fibres of $p_{X'} \times p_{X'}$ have dimension $2g$, Theorem 9.2 of *loc.cit.* now implies $(p_{X'} \times p_{X'})_*(j \times j)^*(\tilde{\Pi}_X) \in J^{(1+g)/2}(X_0 \times X_0, \Sigma_0)$.

5 The symplectic structure along the cone.

Let

$$\Sigma_\alpha = \{(x, r\alpha_x) : x \in X, r > 0\} \subseteq T^*(X) \setminus \{0\} \quad (21)$$

be the half-line bundle generated by the connection 1-form. This is a closed symplectic cone in $T^*(X)$. Let $\rho : \Sigma_\alpha \rightarrow X$ and $q = \pi \circ \rho : \Sigma_\alpha \rightarrow M$ be the projections. Since Σ_α is naturally diffeomorphic to $X \times \mathbb{R}_+$, we have an intrinsic isomorphism of vector bundles on Σ_α ,

$$T(\Sigma_\alpha) \cong \rho^*(TX) \oplus \text{span} \left\{ \frac{\partial}{\partial r} \right\},$$

where $\frac{\partial}{\partial r}$ denotes the generator of the \mathbb{R}_+ -action on Σ_α . On the other hand, the connection 1-form induces a splitting $T(X) = \pi^*(T(M)) \oplus \text{span} \left\{ \frac{\partial}{\partial \theta} \right\}$, where $\frac{\partial}{\partial \theta}$ denotes the generator of the S^1 -action. On the upshot, we have an isomorphism of vector bundles on Σ_α :

$$T(\Sigma_\alpha) \cong q^*(T(M)) \oplus \text{span} \left\{ \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right\}. \quad (22)$$

Since X is oriented, it is metilinear and therefore $T^*(X)$ is metaplectic. Thus $T(T^*(X))$ is a metaplectic vector bundle, and therefore so is its restriction $T(T^*(X))|_{\Sigma_\alpha}$. We shall now examine the situation in local coordinates along Σ_α .

Fix $z = (x, r\alpha_x) \in \Sigma_\alpha$ and set $p = \pi(x)$. Let $U \subseteq X$ be a coordinate neighbourhood for X near x . Thus $T^*U \subseteq T^*X$ is an open subset, and there is an obvious symplectomorphism $T^*U \cong U \times \mathbb{R}_{2n+1}$, where the latter is viewed as an open subset of $\mathbb{R}^{2n+1} \times \mathbb{R}_{2n+1}$, with its natural symplectic structure (here $\mathbb{R}_{2n+1} = (\mathbb{R}^{2n+1})^*$). We shall identify \mathbb{R}_{2n+1} with \mathbb{R}^{2n+1} by means of the standard scalar product, and view $\alpha|_U$ as a smooth \mathbb{R}^{2n+1} -valued form on U . Let $\text{Jac}_y(\alpha)$ be its Jacobian matrix at $y \in U$.

A differential 2-form on U is represented by smooth function $U \rightarrow A_{2n+1}$, where A_{2n+1} is the space of antisymmetric matrices of order $2n+1$. Namely, if we denote by t_i the local coordinates on U , the differential form $\nu = \sum_{i,j} \nu_{ij} dt_i \wedge dt_j$ is represented by the A_{2n+1} -valued function $[\nu_{i,j}]$. The proof of the following is left to the reader:

Lemma 16. *The matrix valued function $\frac{1}{2} [\text{Jac}(\alpha)^t - \text{Jac}(\alpha)]$ represents $\pi^*(\Omega)$ on U .*

In other words, if $y \in U$ and $v = \sum_i v_i \frac{\partial}{\partial t_i} \Big|_y$, $w = \sum_i w_i \frac{\partial}{\partial t_i} \Big|_y \in T_y(X)$ then

$$\pi^*(\Omega)_y(v, w) = \frac{1}{2} v^t [\text{Jac}_y(\alpha)^t - \text{Jac}_y(\alpha)] w.$$

Corollary 8. *Let $V = \sum_i V_i \frac{\partial}{\partial t_i}$ be a vector field on U . Then the contraction $\iota(V)\pi^*(\Omega) = \pi^*(\Omega)(V, \cdot)$ is the 1-form represented on U by the vector valued function $\frac{1}{2} [\text{Jac}_y(\alpha) - \text{Jac}_y(\alpha)^t] V$.*

Lemma 17. *The pull-back $q^*(TM)$ has a natural metaplectic structure.*

By (22), Lemma 17 implies that Σ has a natural metaplectic structure.

Proof. With some abuse of language, in the symplectic coordinate chart $U \times \mathbb{R}^{2n+1}$ the tangent space $T_z(\Sigma_\alpha)$ is the symplectic subspace of $\mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1}$ given by

$$\begin{aligned} T_z(\Sigma_\alpha) &= \left\{ \begin{pmatrix} v \\ r \text{Jac}_x(\alpha)v \end{pmatrix} : v \in \mathbb{R}^{2n+1} \right\} \oplus \text{span} \left\{ \begin{pmatrix} 0 \\ \alpha_x \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} v \\ r \text{Jac}_x(\alpha)v \end{pmatrix} : v \in \mathbb{R}^{2n+1}, \alpha_x(v) = 0 \right\} \\ &\quad \oplus \text{span} \left\{ \begin{pmatrix} \kappa \\ r \text{Jac}_x(\alpha)\kappa \end{pmatrix}, \begin{pmatrix} 0 \\ \alpha_x \end{pmatrix} \right\}, \end{aligned} \quad (23)$$

where κ represents $\frac{\partial}{\partial \theta}$. Clearly, (23) is just (22) in local coordinates. It follows that the symplectic orthocomplement of $T_z(\Sigma_\alpha)$ in $T_z(T^*(X))$ is given by

$$T_z(\Sigma_\alpha)^\perp = \left\{ \begin{pmatrix} v \\ r \text{Jac}_x^t(\alpha)v \end{pmatrix} : v \in \ker(\alpha_x) \subseteq \mathbb{R}^{2n+1} \right\}. \quad (24)$$

For simplicity, let us use $q^*(TM)_{\text{hom}}$ as a short hand for the symplectic vector bundle $(q^*(TM), 2r q^*(\Omega))$ ($r > 0$ is the conic coordinate on Σ_α). Then (24) shows that the differential dq induces a symplectic isomorphism between the vector subbundle $T(\Sigma_\alpha)^\perp$ and the pull-back $q^*(T(M))_{\text{hom}}$ (i.e., $q^*(TM)_{\text{hom}}$ with the opposite symplectic structure). On the other hand, rescaling provides a symplectic isomorphism $q^*(TM)_{\text{hom}} \cong q^*(TM)$ (the latter as a short hand for $(q^*(TM), q^*(\Omega))$). On the upshot, we have a symplectic isomorphism

$$\begin{aligned} T(T^*X)|_{\Sigma_\alpha} &\cong q^*(TM)_{\text{hom}} \oplus q^*(TM)_{\text{hom}}^- \oplus \text{span} \left\{ \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right\} \\ &\cong q^*(TM) \oplus q^*(TM)^- \oplus \text{span} \left\{ \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right\}. \end{aligned} \quad (25)$$

Let now $\text{Bp}(q^*(TM)) = q^*(\text{Bp}(TM))$ and $\text{Bp}(q^*(TM)^-)$ be the principal $\text{Sp}(n)$ -bundles of all symplectic frames in $q^*(T^*(M))$ and $q^*(T^*(M))^-$, respectively. Let the automorphism $\sigma : \text{Sp}(n) \rightarrow \text{Sp}(n)$ be defined by

$$\sigma(U) =: \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} U \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \quad (U \in \text{Sp}(n)).$$

Let us redefine the action of $\mathrm{Sp}(n)$ on $\mathrm{Bp}(q^*(TM)^-)$ by composing with σ . Then the map $\tau : \mathrm{Bp}(q^*(TM)) \rightarrow \mathrm{Bp}(q^*(TM)^-)$ given by

$$\tau : (e_1, \dots, e_n, f_1, \dots, f_n) \mapsto (-e_1, \dots, -e_n, f_1, \dots, f_n)$$

is an $\mathrm{Sp}(n)$ -equivariant diffeomorphism. We then have an $\mathrm{Sp}(n)$ -equivariant embedding $\hat{\tau} : \mathrm{Bp}(q^*(TM)) \rightarrow \mathrm{Bp}(T^*X|_{\Sigma_\alpha})$ given by

$$(\mathbf{e}, \mathbf{f}) \mapsto \left(\mathbf{e}, \mathbf{f}, -\mathbf{e}, \mathbf{f}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial r} \right),$$

for $(\mathbf{e}, \mathbf{f}) = (e_1, \dots, e_n, f_1, \dots, f_n) \in \mathrm{Bp}(q^*(TM))$. The inverse image of $\hat{\tau} : \mathrm{Bp}(q^*(TM))$ in the metaplectic cover of $\mathrm{Bp}(T^*X|_{\Sigma_\alpha})$ is the asserted metaplectic structure of $q^*(TM)$.

Let us now fix $\zeta = (x, x, r\alpha_x, -r\alpha_x) \in \Sigma$ and examine the symplectic structure of $T^*(X \times X)$ near ζ . We shall also identify $T^*(X \times X)$ with $T^*(X) \times T^*(X)$. In the symplectic product coordinate chart $(U \times \mathbb{R}^{2n+1}) \times (U \times \mathbb{R}^{2n+1})$ the tangent space $T_\zeta(\Sigma)$ is the isotropic subspace of $(\mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1}) \times (\mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1})$ given by

$$T_\zeta(\Sigma) = \left\{ \begin{pmatrix} v \\ r\mathrm{Jac}_x(\alpha)v \\ v \\ -rJ_x(\alpha)v \end{pmatrix} : v \in \mathbb{R}^{2n+1} \right\} \oplus \mathrm{span} \left\{ \begin{pmatrix} 0 \\ \alpha_x \\ 0 \\ -\alpha_x \end{pmatrix} \right\}.$$

The symplectic annihilator of $T_\zeta(\Sigma)$ is then

$$T_\zeta(\Sigma)^\perp = \left\{ \begin{pmatrix} v \\ r\mathrm{Jac}_x(\alpha)^t v \\ w \\ -r\mathrm{Jac}_x(\alpha)^t w \end{pmatrix} : \alpha_x(v) = \alpha_x(w) = 0 \right\} + T_\zeta(\Sigma). \quad (26)$$

Lemma 18. *The sum of vector spaces on the left hand side of (26) is direct.*

Proof. Let $a, b, c \in \mathbb{R}$, $v' \in \mathbb{R}^{2n+1}$ and $v, w \in \ker(\alpha_x) \subseteq \mathbb{R}^{2n+1}$ be such that

$$a \begin{pmatrix} v' \\ r\mathrm{Jac}_x(\alpha)v' \\ v' \\ -r\mathrm{Jac}_x(\alpha)v' \end{pmatrix} + b \begin{pmatrix} v \\ r\mathrm{Jac}_x(\alpha)^t v \\ w \\ -r\mathrm{Jac}_x(\alpha)^t w \end{pmatrix} + c \begin{pmatrix} 0 \\ \alpha_x \\ 0 \\ -\alpha_x \end{pmatrix} = 0.$$

If a or b vanish then so do all the other coefficients. We may otherwise absorb them into v and w so as to assume $a = -b = 1$; then $v' = v = w \in \ker(\alpha_x)$. Thus we are reduced to the equality

$$(\text{Jac}_x(\alpha) - \text{Jac}_x(\alpha)^t) v = -\frac{c}{r} \alpha_x,$$

for a certain $v \in \ker(\alpha_x)$. Now $\ker(\alpha_x)$ is the horizontal subspace for the connection, and the skew matrix $\frac{1}{2} [\text{Jac}_x(\alpha)^t - \text{Jac}_x(\alpha)]$ represents the 2-form $d_x \alpha = \pi^*(\Omega)_x$.

Claim 5.1. Let μ be a 2-form on M and let W be a vector field on M . Denote by \tilde{W} the horizontal lift of W to a vector field on X , under the given connection. Let ι be the contraction operator between vector fields and differential forms. Then

$$\iota(\tilde{W}) \pi^*(\mu) = \pi^*(\iota(W) \mu).$$

In fact, both 1-forms vanish on vertical tangent vectors, and they obviously take the same values on horizontal vectors.

Thus, if v represents (in our coordinate patch) the horizontal lift of a tangent vector $\xi \in T_p(M)$, $p = \pi(x)$, then $\frac{1}{2} [\text{Jac}_x(\alpha)^t - \text{Jac}_x(\alpha)] v$ represents the pull-back of the 1-form $\iota(\xi) \Omega_p$ under the differential $d_p \pi : T_x(X) \rightarrow T_p(M)$. But this may not be a multiple of the connection, unless it vanishes. Thus, $c = 0$ and $(\text{Jac}_x(\alpha) - \text{Jac}_x(\alpha)^t) v = 0$, that is, $\iota(\xi) \Omega_p = 0$. This contradicts the nondegeneracy of Ω , unless $\xi = 0$ and therefore $v = 0$. Lemma 18 follows.

Therefore, the vector subspace

$$\left\{ \begin{pmatrix} v \\ r \text{Jac}_x(\alpha)^t v \\ w \\ -r \text{Jac}_x(\alpha)^t w \end{pmatrix} : v, w \in \ker(\alpha_x) \right\} \subseteq T_\zeta(T^*(X) \times T^*(X))$$

is symplectomorphic to the symplectic normal space to Σ at ζ , $N_{\Sigma, \zeta} = (T_\zeta \Sigma)^\perp / T_\zeta \Sigma$. We shall denote by Ω_{can} the canonical symplectic structure of the cotangent bundles $T^*(X)$ and $T^*(X \times X) \cong T^*(X) \times T^*(X)$ at $(x, r\alpha_x)$ and at ζ , respectively. In our coordinate chart, this is the canonical symplectic structure on $(\mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1})$ and $(\mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1}) \times (\mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1})$.

We have, for $v, w, v', w' \in \ker(\alpha_x)$ horizontal lifts of $\xi, \eta, \xi', \eta' \in T_p M$:

$$\begin{aligned} & \Omega_{\text{can}} \left(\begin{pmatrix} v \\ r \text{Jac}_x(\alpha)^t v \\ w \\ -r \text{Jac}_x(\alpha)^t w \end{pmatrix}, \begin{pmatrix} v' \\ r \text{Jac}_x(\alpha)^t v' \\ w' \\ -r \text{Jac}_x(\alpha)^t w' \end{pmatrix} \right) = \\ & \Omega_{\text{can}} \left(\begin{pmatrix} v \\ r \text{Jac}_x(\alpha)^t v \end{pmatrix}, \begin{pmatrix} v' \\ r \text{Jac}_x(\alpha)^t v' \end{pmatrix} \right) - \Omega_{\text{can}} \left(\begin{pmatrix} w \\ r \text{Jac}_x(\alpha)^t w \end{pmatrix}, \begin{pmatrix} w' \\ r \text{Jac}_x(\alpha)^t w' \end{pmatrix} \right) \\ & = r\Omega_p(\xi, \xi') - r\Omega_p(\eta, \eta'). \end{aligned} \quad (27)$$

Thus, there is a natural symplectic isomorphism

$$N_{\Sigma, \zeta} \cong (T_p M, 2r\Omega_p) \oplus (T_p M, -2r\Omega_p), \quad (28)$$

which extends to an isomorphism of symplectic vector bundles with the appropriate symplectic structures. Let J_M be the complex structure of M . Then, when endowed with the compatible complex structure $(J_M, -J_M)$, N_Σ is a unitary (hence Riemannian) vector bundle.

Let us next consider the vector subspace $V_\zeta \subseteq T_\zeta(T^*(X \times X)) \setminus \{0\}$ given by:

$$V_\zeta =: \text{span} \left\{ \begin{pmatrix} v \\ r \text{Jac}_x(\alpha)v \\ w \\ -r \text{Jac}_x(\alpha)w \end{pmatrix} : \alpha_x(v) = \alpha(w) = 0 \right\}.$$

Then V_ζ is also naturally symplectomorphic to $(T_p M, -2r\Omega_p) \oplus (T_p M, 2r\Omega_p)$, $p = \pi(x)$, and thus becomes a unitary vector space with the complex structure $(-J_p, J_p)$; it contains $T_\zeta(\Sigma)$ as a Lagrangian subspace. Clearly, it naturally extends to a vector subbundle of $V \subseteq T(T^*(X) \times T^*(X))$.

Next, let $\kappa_\theta : U \rightarrow \mathbb{R}^{2n+1}$ represent the vector field $\frac{\partial}{\partial \theta}$ generating of the S^1 -action, and let us set $\kappa = \kappa_\theta(x)$. Let us then introduce the vector subspace $W_\zeta \subseteq T_\zeta(T^*(X) \times T^*(X)) \setminus \{0\}$ given by

$$W_\zeta = \text{span} \left\{ \begin{pmatrix} \kappa \\ r \text{Jac}_x(\alpha)\kappa \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ r\alpha \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -r\alpha \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \kappa \\ -r \text{Jac}_x(\alpha)\kappa \end{pmatrix} \right\}.$$

Since $\kappa^t \cdot \alpha_x = \alpha_x \left(\frac{\partial}{\partial \theta} \Big|_x \right) = 1$, by mapping this (intrinsically defined) basis to the real basis (e_1, ie_1, e_2, ie_2) of \mathbb{C}^2 we see that W_ζ is naturally symplectomorphic to $(\mathbb{C}^2, r\omega_0)$, where $\omega_0 = \frac{i}{2} \sum_{i=1}^2 dz_i \wedge d\bar{z}_i$ is the standard symplectic structure. We shall then consider W_ζ as the fibre of a unitary vector bundle W on Σ . The proof of following is left to the reader:

Lemma 19. *We have the symplectic direct sum decomposition*

$$T_\zeta(T^*(X) \times T^*(X)) \cong N_{\Sigma, \zeta} \oplus V_\zeta \oplus W_\zeta.$$

We may then take the direct sum of the unitary structures on each summand to make $T(T^*(X) \times T^*(X))|_\Sigma$ into a unitary vector bundle over Σ .

We next consider the closed isotropic cone

$$\Sigma' = \{(x, r\alpha'_x, x, -r\alpha'_x) : x \in X', r > 0\} \subseteq T^*(X') \times T^*(X') \setminus \{0\},$$

where $\alpha' = j^*(\alpha)$. Let us fix $\zeta' = (x, r\alpha'_x, x, -r\alpha'_x) \in \Sigma'$, and choose a coordinate patch $U' \subseteq X'$ containing x . Thus $T^*(U') \cong U' \times \mathbb{R}^{2n+1-g}$, and α' is represented locally near x by a smooth function $U' \rightarrow \mathbb{R}^{2n+1-g}$. Let $\text{Jac}'_x(\alpha')$ be its Jacobian matrix at x . Then in local coordinates

$$T_{\zeta'}(\Sigma') = \left\{ \begin{pmatrix} v \\ r\text{Jac}'_x(\alpha')v \\ v \\ -r\text{Jac}'_x(\alpha')v \end{pmatrix} : v \in \mathbb{R}^{2n+1-g} \right\} \oplus \text{span} \left\{ \begin{pmatrix} 0 \\ \alpha'_x \\ 0 \\ -\alpha'_x \end{pmatrix} \right\}.$$

The symplectic annihilator is then $T_{\zeta'}(\Sigma')^\perp = R_{\zeta'} + T_{\zeta'}(\Sigma')$, where

$$\begin{aligned} R_{\zeta'} = & \left\{ \begin{pmatrix} v \\ r\text{Jac}'_x(\alpha')^t v \\ w \\ -r\text{Jac}'_x(\alpha')^t w \end{pmatrix} : v, w \in \ker(\alpha'_x) \right\} \\ & \oplus \text{span} \left\{ \begin{pmatrix} 0 \\ g_{\xi, x} \\ 0 \\ -g_{\xi, x} \end{pmatrix} : \xi \in \mathfrak{g} \right\}. \end{aligned} \quad (29)$$

Here, for every $\xi \in \mathfrak{g}$, $g_{\xi, x}$ denotes the linear functional on $T_x(X')$ given by the Riemannian scalar product with $\xi^\sharp(x)$, where ξ^\sharp is the vector field on X' generated by ξ . Because of the degeneracy of the restricted form, the two summands for $T_{\zeta'}(\Sigma')^\perp$ have a g -dimensional intersection:

Lemma 20.

$$R_{\zeta'} \cap T_{\zeta'}(\Sigma') = \left\{ \begin{pmatrix} \xi^\sharp(x) \\ r\text{Jac}'_x(\alpha')\xi^\sharp(x) \\ \xi^\sharp(x) \\ -r\text{Jac}'_x(\alpha')\xi^\sharp(x) \end{pmatrix} : \xi \in \mathfrak{g} \right\}.$$

Proof. Notice to begin with that the action of G on X is horizontal on X' , that is, $\xi^\sharp(x) \in \ker(\alpha'_x)$ if $\xi \in \mathfrak{g}$ and $x \in X'$. Let $\Omega' = \iota^*(\Omega)$, where $\iota : M' \hookrightarrow M$ is the inclusion. Then Ω' is a degenerate closed 2-form, whose kernel at any $p \in M'$ is the tangent space $\mathfrak{g} \cdot p \subseteq T_p(M)$ to the orbit through p . On the other hand, $\pi'^*(\Omega') = d\alpha'$, where $\pi' : X' \rightarrow M'$ is the projection. Since the space of all $\xi^\sharp(x)$'s is the horizontal lift of $\mathfrak{g} \cdot p$ at x , we deduce that in our local coordinates we have $\xi^\sharp(x) \in \ker(\text{Jac}'_x(\alpha') - \text{Jac}'_x(\alpha')^t)$ for all $\xi \in \mathfrak{g}$. On the upshot, $\ker(\alpha'_x) \cap \ker(\text{Jac}'_x(\alpha') - \text{Jac}'_x(\alpha')^t)$ is precisely the space of all $\xi^\sharp(x)$'s, $\xi \in \mathfrak{g}$. Suppose now that we have an equality:

$$\begin{pmatrix} v' \\ r\text{Jac}'_x(\alpha')v' \\ v' \\ -r\text{Jac}'_x(\alpha')v' \end{pmatrix} + b \begin{pmatrix} 0 \\ \alpha'_x \\ 0 \\ -\alpha'_x \end{pmatrix} = \begin{pmatrix} v \\ r\text{Jac}'_x(\alpha')^t v \\ w \\ -r\text{Jac}'_x(\alpha')^t w \end{pmatrix} + \begin{pmatrix} 0 \\ g_{\xi,x} \\ 0 \\ -g_{\xi,x} \end{pmatrix},$$

with $v, w \in \ker(\alpha'_x)$ and $\xi \in \mathfrak{g}$. Then $v' = v = w \in \ker(\alpha'_x)$ and so:

$$r(\text{Jac}'_x(\alpha') - \text{Jac}'_x(\alpha')^t)v + b\alpha'_x - g_{\xi,x} = 0.$$

The left hand side is a cotangent vector to X' at x . By pairing this first with the generator at x of the S^1 -action on X' , and then with the $\eta^\sharp(x)$'s, $\eta \in \mathfrak{g}$, we obtain $b = 0$ and $\xi = 0$. Thus, $v = \xi^\sharp(x)$ for some $\xi \in \mathfrak{g}$.

Let $H(M'/M_0) \subseteq TM'$ be the Riemannian orthocomplement of the vertical tangent bundle of $p_{M'} : M' \rightarrow M_0$. Then $H(M'/M_0)$ is a connection for the principal G -bundle $p_{M'}$. When M' is endowed with the 2-form $\Omega' = \iota^*(\Omega) = p_{M'}^*(\Omega_0)$, $H(M'/M_0)$ is a symplectic vector subbundle of TM' , symplectomorphic to the pull-back $p_{M'}^*(TM_0)$. Let $\Omega_{H(M'/M_0)}$ be its symplectic structure. Now R_ζ in (29) may be decomposed as

$$\begin{aligned} R_{\zeta'} = & \left\{ \begin{pmatrix} v \\ r\text{Jac}'_x(\alpha')^t v \\ w \\ -r\text{Jac}'_x(\alpha')^t w \end{pmatrix} : v, w \in \ker(\alpha'_x), d_x\pi(v), d_x\pi(w) \in H_p \right\} \\ & \oplus \left\{ \begin{pmatrix} \xi^\sharp(x) \\ r\text{Jac}'_x(\alpha')^t \xi^\sharp(x) \\ -\xi^\sharp(x) \\ r\text{Jac}'_x(\alpha')^t \xi^\sharp(x) \end{pmatrix} : \xi \in \mathfrak{g} \right\} \oplus \text{span} \left\{ \begin{pmatrix} 0 \\ g_{\xi,x} \\ 0 \\ -g_{\xi,x} \end{pmatrix} : \xi \in \mathfrak{g} \right\} \\ & \oplus \left\{ \begin{pmatrix} \xi^\sharp(x) \\ r\text{Jac}'_x(\alpha')^t \xi^\sharp(x) \\ \xi^\sharp(x) \\ -r\text{Jac}'_x(\alpha')^t \xi^\sharp(x) \end{pmatrix} : \xi \in \mathfrak{g} \right\}. \end{aligned} \tag{30}$$

By Lemma 20, the first two summands add up symplectomorphically to the symplectic normal bundle to Σ' in $T^*(X') \times T^*(X')$. Thus,

Lemma 21. *Let $N'_{\Sigma'}$ be the symplectic normal bundle of $\Sigma' \subseteq T^*(X') \times T^*(X')$. Let $\zeta' = (x, r\alpha'_{x'}, x, -r\alpha'_{x'})$, $p = \pi(x) \in M$, $\bar{p} = q(p) \in M_0$. Then we have a natural isomorphism of symplectic vector spaces:*

$$\begin{aligned} N'_{\Sigma', \zeta'} &= (H_p(M'/M_0), 2r\Omega_{H(M'/M_0), p}) \oplus (H_p(M'/M_0), -2r\Omega_{H(M'/M_0), p}) \oplus \mathfrak{g}_{\mathbb{C}, \bar{p}} \\ &\cong (T_{\bar{p}}(M_0), r\Omega_{0, \bar{p}}) \oplus (T_{\bar{p}}(M_0), -r\Omega_{0, \bar{p}}) \oplus \mathfrak{g}_{\mathbb{C}, p}. \end{aligned}$$

This naturally extends to an isomorphism of unitary vector bundles over Σ' .

Here,

$$\mathfrak{g}_{\mathbb{C}, p} = \text{span} \left\{ \begin{pmatrix} \xi^\sharp(x) \\ r\text{Jac}'_x(\alpha')\xi^\sharp(x) \\ -\xi^\sharp(x) \\ r\text{Jac}'_x(\alpha')\xi^\sharp(x) \end{pmatrix}, \begin{pmatrix} 0 \\ g_{\xi, x} \\ 0 \\ -g_{\xi, x} \end{pmatrix} : \xi \in \mathfrak{g} \right\}.$$

On $T_p(M) \supseteq T_p(M')$, we have in obvious notation $g_\xi = \Omega_p(\cdot, J_p \xi^\sharp(p))$. Thus, $\mathfrak{g}_{\mathbb{C}, p}$ may be naturally identified with the complexified Lie algebra of G , that is, the Lie algebra of the complexified group \tilde{G} . It is endowed with the unitary structure induced by its infinitesimal action on M at p , which makes it into a complex subspace of $T_p(M)$: $\mathfrak{g}_{\mathbb{C}, p} \cong \mathfrak{g}_{\mathbb{C}} \cdot p = T_p(\tilde{G} \cdot p)$. Let κ'_θ denote the generator of the S^1 -action on X' , set $\kappa' = \kappa'_\theta(x)$ and let us now introduce the vector bundles over Σ' by setting, in local coordinates,

$$\begin{aligned} V'_{\zeta'} &= \left\{ \begin{pmatrix} v \\ r\text{Jac}_x(\alpha')v \\ w \\ -r\text{Jac}_x(\alpha')w \end{pmatrix} : v, w \in \ker(\alpha'_x), d_x\pi(v), d_p\pi(w) \in H_p \right\}, \\ V''_{\zeta'} &= \left\{ \begin{pmatrix} \xi^\sharp(x) \\ r\text{Jac}_x(\alpha')\xi^\sharp(x) \\ \xi^\sharp(x) \\ -r\text{Jac}_x(\alpha')\xi^\sharp(x) \end{pmatrix}, \begin{pmatrix} 0 \\ g_\xi \\ 0 \\ g_\xi \end{pmatrix} : \xi \in \mathfrak{g} \right\}, \\ W'_{\zeta'} &= \text{span} \left\{ \begin{pmatrix} \kappa' \\ r\text{Jac}'_x(\alpha')\kappa' \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ r\alpha'_x \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -r\alpha'_x \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \kappa' \\ -r\text{Jac}'_x(\alpha')\kappa' \end{pmatrix} \right\}. \end{aligned}$$

Just as before, these are naturally unitary vector bundles, and we have:

Lemma 22. *There is a symplectic direct sum decomposition*

$$T(T^*(X) \times T^*(X))|_{\Sigma'} \cong N'_{\Sigma'} \oplus V' \oplus V'' \oplus W'.$$

By taking the direct sum of the unitary structure of each summand, this makes $T(T^*(X) \times T^*(X))|_{\Sigma'}$ into a unitary vector bundle.

6 The symbol of $(j \times j)^* (\tilde{\Pi}_X)$.

If W is a manifold, the symbol of a Fourier-Hermite distribution $u \in J^k(W, \Xi)$ associated to a closed isotropic cone $\Xi \subseteq T^*W \setminus \{0\}$ is a smooth section of the symplectic spinor bundle, $\text{Spin}(\Xi)$, homogeneous of degree k with respect to the conic structure of Ξ . The space of homogeneous sections of degree k of $\text{Spin}(\Xi)$ will be denoted by $S^k(\Xi)$.

In particular, the symbol of the Szegő kernel $\tilde{\Pi}_X \in J^{1/2}(X \times X, \Sigma)$ is an element of $S^{1/2}(\Sigma)$. The spinor bundle of $\Sigma \subseteq T^*(X \times X) \setminus \{0\}$ is $\text{Spin}(\Sigma) = \bigwedge^{1/2} \Sigma \otimes S(N_\Sigma)$, where N_Σ is the symplectic normal bundle of Σ . Now Σ is obviously diffeomorphic to Σ_α in (21), and by the symplecticity of Ω the latter is a symplectic submanifold of T^*X .

Definition 1. By the above, Σ carries a built-in symplectic structure homogeneous of degree one. We shall denote by $\text{vol}_\Sigma^{1/2}$ the nowhere vanishing half-form of degree $1/2$ obtained from the latter by the appropriate homogenization with respect to the r coordinate.

On the other hand, in view of (28), the symplectic normal bundle of Σ is (after an appropriate rescaling) naturally isomorphic to $q^*(TM) \oplus q^*(TM)^\perp$. Thus, given the complex structure J_M of the base manifold M , by Corollary 6 the bundle $\mathcal{S}(N_\Sigma) \in \text{End}_{\mathcal{HS}}(\mathcal{S}(q^*(TM)))$ has a built-in nowhere vanishing section σ_{J_M} .

After [BG], the symbol of the Szegő kernel is the tensor product

$$\sigma(\tilde{\Pi}_X) = \text{vol}_\Sigma^{1/2} \otimes \sigma_{J_M}. \quad (31)$$

As an intermediate step towards computing the symbol of $\tilde{\Pi}_{X|X_0}$, we shall now compute the symbol of the restriction $(j \times j)^* (\tilde{\Pi}_X)$.

If $x \in X' \subseteq X$ and $\eta \in T_x^*X$ is a cotangent vector to X at x , we shall use the notation $\eta' = (d_x j)^*(\eta) \in T_x^*(X')$ for the restriction of η to X' . The relevant canonical relation in $T^*(X' \times X') \times T^*(X \times X)$ is thus

$$\Gamma_{j \times j} = \left\{ ((x_1, \eta'_1, x_2, \eta'_2), (x_1, \eta_1, x_2, \eta_2)) : x_i \in X', \eta_i \in T_{x_i}^*(X) \right\},$$

and the fibre product $F_{j \times j}$ of Σ and $\Gamma_{j \times j}$ maps diffeomorphically to Σ' under the projection $p_{F_{j \times j}} : F_{j \times j} \rightarrow \Gamma_{j \times j} \circ \Sigma = \Sigma'$. Let $q : T^*(X' \times X') \times T^*(X \times X) \rightarrow T^*(X \times X)$ be the projection onto the second factor. Fix $\zeta = (x, r\alpha, x, -r\alpha) \in \Sigma$; the differential of q at (ζ', ζ) ,

$$q_\zeta =: d_{(\zeta', \zeta)} q : T_{\zeta'}(T^*(X' \times X')) \times T_\zeta(T^*(X \times X)) \rightarrow T_\zeta(T^*(X \times X)),$$

is simply projection onto the second factor. Then we have (in local coordinates)

$$q_\zeta \left(T_{(\zeta', \zeta)}(\Gamma_{J \times J}) \right) = \left\{ \begin{pmatrix} v \\ \phi \\ w \\ \eta \end{pmatrix} : v, w \in T_x(X'), \phi, \eta \in \mathbb{R}^{2n+1} \right\}.$$

Given that $X' = (\Phi \circ \pi)^{-1}(0)$, its tangent space $T_x(X')$ is defined by the vanishing of all the differentials $d_x H_\xi$, $\xi \in \mathfrak{g}$, where $H_\xi = \langle \xi, \Phi \rangle$ is the Hamiltonian function associated to ξ . Its symplectic annihilator in $T_\zeta(T^*(X) \times T^*(X))$ is then

$$q_\zeta \left(T_{(\zeta', \zeta)}(\Gamma_{J \times J}) \right)^\perp = \left\{ \begin{pmatrix} 0 \\ d_x H_\xi \\ 0 \\ d_x H_\eta \end{pmatrix} : \xi, \eta \in \mathfrak{g} \right\}.$$

It follows that $U_{0\zeta} =: q_\zeta \left(T_{(\zeta', \zeta)}(\Gamma_{J \times J}) \right)^\perp \cap T_\zeta(\Sigma) = \{0\}$, while

$$U_{1\zeta} =: q_\zeta \left(T_{(\zeta', \zeta)}(\Gamma_{J \times J}) \right)^\perp \cap T_\zeta(\Sigma)^\perp = \text{span} \left\{ \begin{pmatrix} 0 \\ d_x H_\xi \\ 0 \\ -d_x H_\xi \end{pmatrix} : \xi \in \mathfrak{g} \right\}.$$

Thus, in view of the equality holding in local coordinates on $U \subseteq X$ (see Claim 5.1)

$$dH_\xi = \iota(X_\xi)\Omega = \frac{1}{2} (\text{Jac}(\alpha) - \text{Jac}(\alpha)^t)X_\xi \quad (\xi \in \mathfrak{g}),$$

the vector space $U_{1\zeta} \cong U_\zeta \subseteq N_{\Sigma, \zeta} \cong (T_p M, 2r\Omega_p) \oplus (T_p M, -2r\Omega_p)$ may be identified with the isotropic subspace

$$U_p \cong \{(X_\xi(p), X_\xi(p)) : \xi \in \mathfrak{g}\}, \quad (32)$$

where X_ξ is the vector field generated by $\xi \in \mathfrak{g}$. As above, we shall make $(T_p M, 2r\Omega_p) \oplus (T_p M, -2r\Omega_p)$ into a unitary vector space, by endowing it with the compatible complex structure $(J_p, -J_p)$. Furthermore, let \mathfrak{g}^M denote the trivial vector subbundle of $TM|_{M'}$ on M' with fibre \mathfrak{g} generated by the infinitesimal action of \mathfrak{g} on M . Then \mathfrak{g}^M is an oriented isotropic subbundle of $TM|_{M'}$. We may write (32) as

$$U = \text{diag}(\mathfrak{g}^M) \subseteq (T_p M, 2r\Omega_p) \oplus (T_p M, -2r\Omega_p). \quad (33)$$

Therefore, Corollary 7 may be applied to the symplectic normal bundle N_U of U in $(T_p M, 2r\Omega_p) \oplus (T_p M, -2r\Omega_p)$. To this end, let us adopt the notation introduced in the discussion preceding (30) and in Corollary 7 with $L = \mathfrak{g}^M$, and recall after [GS2] that the symplectic and Riemannian annihilators of \mathfrak{g}^M within $TM|_{M'}$ are given by $(\mathfrak{g}^M)^\perp = \mathfrak{g}^M \oplus H(M'/M_0)$ and $(\mathfrak{g}^M)^0 = J_M(\mathfrak{g}^M) \oplus H(M'/M_0)$, respectively.

Let us then define vector bundles \mathfrak{g}_r^M and \mathfrak{g}_i^M on M' by setting $\mathfrak{g}_r^M(p) =: \{(X_\xi(p), -X_\xi(p)) : \xi \in \mathfrak{g}\}$ and $\mathfrak{g}_i^M(p) =: \{(J_p X_\xi(p), J_p X_\xi(p)) : \xi \in \mathfrak{g}\}$ ($p \in M'$), and then set $(\mathfrak{g}^M)_\mathbb{C} = \mathfrak{g}_r^M \oplus \mathfrak{g}_i^M$.

By Corollary 7 iii), we conclude

Corollary 9. *the symplectic annihilator of U in N_Σ is*

$$N_U \cong (\mathfrak{g}^M)_\mathbb{C} \oplus (q^*H(M'/M_0), 2r\Omega_{H(M/M')}) \oplus (q^*H(M'/M_0), -2r\Omega_{H(M/M')}). \quad (34)$$

Lemma 23. *N_U is a metaplectic vector bundle.*

Proof. Let us consider the nested fibre diagrams

$$\begin{array}{ccccccc} \Sigma' & \xrightarrow{\rho'} & X' & \xrightarrow{\pi_{X'}} & M' & & \\ p_{\Sigma'} \downarrow & & p_{X'} \downarrow & & \downarrow & & p_{M'} \\ \Sigma_0 & \xrightarrow{\rho_0} & X_0 & \xrightarrow{\pi_{X_0}} & M_0. & & \end{array} \quad (35)$$

The horizontal arrows are principal S^1 -bundles, and the vertical ones are principal G -bundles. Set $q' = \pi_{X'} \circ \rho' : \Sigma' \rightarrow M'$, $q_0 = \pi_{X_0} \circ \rho_0 : \Sigma_0 \rightarrow M_0$. It suffices to show that $q'^*(H(M'/M_0))$ is a metaplectic vector bundle, and thus by commutativity that $q_0^*(TM_0)$ is metaplectic. This follows from Lemma 17 applied to q_0 .

Equivalently, $q^*(H(M'/M_0))|_{\Sigma'}$ is the symplectic normal bundle of the oriented isotropic subbundle $q^*(\mathfrak{g}^M)|_{\Sigma'}$ in the metaplectic vector bundle $q^*(TM)|_{\Sigma'}$, and the fact that it is metaplectic then follows from Proposition 1.

In view of Lemma 13 and Corollary 6, we have

$$\mathcal{S}(N_U) \cong \mathcal{S}(\mathfrak{g}_\mathbb{C}^M) \otimes \text{End}_{\mathcal{HS}}(\mathcal{S}(q'^*H(M'/M_0))).$$

Given that $(\mathfrak{g}^M)_\mathbb{C}$ is a complex vector subbundle of TM , the complex structure J_M of M singles out the section $\sigma_{\mathfrak{g}^M, J_M}$ of $\mathcal{S}((\mathfrak{g}^M)_\mathbb{C})$. Since $H(M'/M) \cong$

$p_{M'}^*(TM_0)$, the complex structure J_0 of M_0 singles out the section σ_{J_0} of $\text{End}_{\mathcal{HS}}(\mathcal{S}(q'^*H(M'/M_0)))$. Let σ_{J_M} be as in (31) and let $\Phi_U : \mathcal{S}(N_\Sigma) \rightarrow \bigwedge^{-1/2}(U) \otimes \mathcal{S}(N_U)$ be the morphism of vector bundles from Corollary 5. Then, in view of Corollary 7, iv), we have

$$\Phi(\sigma_{J_M}) = \text{vol}^{-1/2}(U) \otimes \sigma_{\mathfrak{g}^M, J_M} \otimes \sigma_{J_0}, \quad (36)$$

where $\text{vol}^{-1/2}(U)$ is the $-\frac{1}{2}$ -form on U taking value one on oriented orthonormal frames of U .

Let now $\text{vol}_{\Gamma_{\tilde{j} \times \tilde{j}}}^{1/2}$ be the half-form on $\Gamma_{\tilde{j} \times \tilde{j}}$ associated to the morphism of metalinear manifolds $\tilde{j} \times \tilde{j}$ as in (5), and let $\text{vol}_\Sigma^{1/2}$ be as in (31). In the following definition, recall that the complex structure J_M maps the vector subbundle \mathfrak{g}^M of $TM|_{M'}$ isometrically onto the vector subbundle \mathfrak{g}^M .

Definition 2. Let ε be the half-form on \mathfrak{g}^M taking value one on oriented orthonormal frames. Given the exact sequence

$$0 \longrightarrow T(\Sigma') \longrightarrow T(\Sigma)|_{\Sigma'} \longrightarrow J_M(\mathfrak{g}^M) \longrightarrow 0,$$

let $\text{vol}_{\Sigma'}^{1/2}$ be the half-form on Σ' , homogeneous of degree $\frac{1}{2}$, obtained by dividing $\text{vol}_\Sigma^{1/2}$ by ε .

Lemma 24. Suppose $x \in X$ and let $\zeta = (x, \alpha_x, x, -\alpha_x)$ (thus, $r = 1$). Then the image at ζ of $\text{vol}_{\Gamma_{\tilde{j} \times \tilde{j}, \zeta}}^{1/2} \otimes \text{vol}_{\Sigma, \zeta}^{1/2} \otimes \text{vol}_{U, \zeta}^{-1/2}$ in $\bigwedge^{1/2}(\Sigma')_\zeta$ under the line bundle isomorphism (14) is $2^g \text{vol}_{\Sigma', \zeta}^{1/2}$.

The factor 2^g is related to the relative dimension of the embedding $\tilde{j} \times \tilde{j}$, which is $2g$. A similar factor of 2^{-g} will appear in the following section, due to the fact that the relative dimension of $p_{X'} \times p_{X'}$ is $-2g$. The two factors will thus cancel out in the final result.

Proof. We shall henceforth omit the base point ζ . Let us fix an oriented orthonormal (real) basis $\{v_j\}_{1 \leq j \leq 2n}$ of $(T_p M, 2r\Omega_p)$, where $p = \pi(x)$, of the following form. First, let us choose an oriented orthonormal basis $\{v_j\}$ of $H_p(M'/M_0)$ for $1 \leq j \leq 2(n-g)$. Next, let us set $v_{2(n-g)+j} = \xi_j^\sharp(p)$, $1 \leq j \leq g$, where $\{\xi_j^\sharp(p)\}$ is any orthonormal frame of $\mathfrak{g}^M(p)$ (the vectors $\xi_j \in \mathfrak{g}$ depend on p), and finally let us set $v_{2n-g+j} = J_{M,p} \xi_j^\sharp(p)$, $1 \leq j \leq g$. In particular, $\{v_j\}_{1 \leq j \leq 2n-g}$ is an oriented orthonormal basis of $T_p M'$. To simplify our expressions, we shall in the following denote by \vec{v} the sequence

$$\left\{ \frac{1}{\sqrt{2}} v_1, \dots, \frac{1}{\sqrt{2}} v_{2(n-g)} \right\},$$

by $\overrightarrow{\xi^\sharp}$ the sequence

$$\left\{ \frac{1}{\sqrt{2}} \xi_1^\sharp(p), \dots, \frac{1}{\sqrt{2}} \xi_g^\sharp(p) \right\},$$

and by $\overrightarrow{\mathbf{J}\xi^\sharp}$ the sequence

$$\left\{ \frac{1}{\sqrt{2}} J_{M,p} X_{\xi_1}(p), \dots, \frac{1}{\sqrt{2}} J_{M,p} \xi_g^\sharp(p) \right\}.$$

With our usual slight abuse of language, we shall identify Σ' with the subset of Σ lying over $\text{diag}(X')$. As in section 5, we shall denote by κ the generator of the S^1 -action on X (expressed in local coordinates). As an oriented orthonormal basis of $T_\zeta(\Sigma')$, we shall then take

$$\mathcal{B}_{\Sigma'} = \left\{ \begin{pmatrix} \mathbf{v} \\ \text{Jac}_x(\alpha)\mathbf{v} \\ \mathbf{v} \\ -\text{Jac}_x(\alpha)\mathbf{v} \end{pmatrix}, \begin{pmatrix} \overrightarrow{\xi^\sharp} \\ \text{Jac}_x(\alpha)\overrightarrow{\xi^\sharp} \\ \overrightarrow{\xi^\sharp} \\ -\text{Jac}_x(\alpha)\overrightarrow{\xi^\sharp} \end{pmatrix}, \begin{pmatrix} \kappa/\sqrt{2} \\ \text{Jac}_x(\alpha)\kappa/\sqrt{2} \\ \kappa/\sqrt{2} \\ -\text{Jac}_x(\alpha)\kappa/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 \\ \alpha_x/\sqrt{2} \\ 0 \\ -\alpha_x/\sqrt{2} \end{pmatrix} \right\}. \quad (37)$$

Let us set $\varphi_{JX} = \frac{1}{2}[\text{Jac}_x(\alpha)^t - \text{Jac}_x(\alpha)]\mathbf{JX}$. We may extend $\mathcal{B}_{\Sigma'}$ to a basis $\mathcal{B}_\Gamma = \mathcal{B}_{\Sigma'} \cup \mathcal{B}_N$ of $T_\zeta(\Gamma_{\tilde{j} \times \tilde{j}})$ by letting

$$\begin{aligned} \mathcal{B}_N = & \left\{ \begin{pmatrix} 0 \\ \alpha_x/\sqrt{2} \\ 0 \\ \alpha_x/\sqrt{2} \end{pmatrix}, \begin{pmatrix} \mathbf{v} \\ \text{Jac}_x(\alpha)^t \mathbf{v} \\ \mathbf{v} \\ -\text{Jac}_x(\alpha)^t \mathbf{v} \end{pmatrix}, \begin{pmatrix} \overrightarrow{\xi^\sharp} \\ \text{Jac}_x(\alpha)^t \overrightarrow{\xi^\sharp} \\ \overrightarrow{\xi^\sharp} \\ -\text{Jac}_x(\alpha)^t \overrightarrow{\xi^\sharp} \end{pmatrix}, \right. \\ & \begin{pmatrix} \mathbf{v} \\ \text{Jac}_x(\alpha)^t \mathbf{v} \\ -\mathbf{v} \\ \text{Jac}_x(\alpha)^t \mathbf{v} \end{pmatrix}, \begin{pmatrix} \mathbf{v} \\ \text{Jac}_x(\alpha)\mathbf{v} \\ -\mathbf{v} \\ \text{Jac}_x(\alpha)\mathbf{v} \end{pmatrix}, \begin{pmatrix} \overrightarrow{\xi^\sharp} \\ \text{Jac}_x(\alpha)\overrightarrow{\xi^\sharp} \\ -\overrightarrow{\xi^\sharp} \\ \text{Jac}_x(\alpha)\overrightarrow{\xi^\sharp} \end{pmatrix}, \begin{pmatrix} \overrightarrow{\xi^\sharp} \\ \text{Jac}_x(\alpha)^t \overrightarrow{\xi^\sharp} \\ -\overrightarrow{\xi^\sharp} \\ \text{Jac}_x(\alpha)^t \overrightarrow{\xi^\sharp} \end{pmatrix}, \\ & \left. \begin{pmatrix} 0 \\ \varphi_{\overrightarrow{\mathbf{J}\xi^\sharp}} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \varphi_{\overrightarrow{\mathbf{J}\xi^\sharp}} \end{pmatrix}, \begin{pmatrix} \kappa/\sqrt{2} \\ \text{Jac}_x(\alpha)\kappa/\sqrt{2} \\ -\kappa \\ \text{Jac}_x(\alpha)\kappa/\sqrt{2} \end{pmatrix} \right\} \end{aligned} \quad (38)$$

Performing appropriate column operations, we obtain a basis whose first $4n+2$ vectors lie in $T_\zeta^*(X \times X)$, while the remaining ones project down to an oriented orthonormal basis in $T_{(x,x)}(X' \times X')$. The statement follows from this and (5).

This holds at any point where $r = 1$. Taking into account the appropriate homogeneity, on the upshot we have proved:

Proposition 3. *The symbol of $(j \times j)^* \left(\tilde{\Pi}_X \right)$ is the section of $\text{Spin}(\Sigma') = \bigwedge^{1/2}(\Sigma') \otimes \mathcal{S}(N_U)$ given by*

$$\sigma \left((j \times j)^* \left(\tilde{\Pi}_X \right) \right) (x, r\alpha'_{x'}, x, -r\alpha'_x) = 2^g r^{g/2} \text{vol}_{\Sigma'}^{1/2} \otimes \sigma_{\mathfrak{g}^M, J} \otimes \sigma_{J_0} \quad (39)$$

$$((x, r\alpha'_x, x, -r\alpha'_x) \in \Sigma').$$

7 The symbol of $\tilde{\Pi}_{X|X_0}$.

We shall now examine the symbol of $\tilde{\Pi}_{X|X_0}$, and relate it to the symbol of the Szegő kernel of X_0 , $\sigma(\tilde{\Pi}_{X_0}) \in S^{1/2}(\Sigma_0)$.

Theorem 2. *The symbol of $\tilde{\Pi}_{X|X_0}$, denoted $\sigma(\tilde{\Pi}_{X|X_0}) \in S^{(1+g)/2}(\Sigma_0)$, is given by*

$$\sigma(\tilde{\Pi}_{X|X_0})(\hat{x}, r\alpha_{0\hat{x}}, \hat{x}, -r\alpha_{0\hat{x}}) = r^{g/2} V_{\text{eff}}(\hat{x}) \sigma(\tilde{\Pi}_{X_0})(\hat{x}, r\alpha_{0\hat{x}}, \hat{x}, -r\alpha_{0\hat{x}})$$

$$(\hat{x} \in X_0, r > 0).$$

Proof. We shall now be more sketchy, since the proof is similar to that of Proposition 3. By homogeneity, it suffices to prove the Theorem for $r = 1$.

Let as above $p_{X'} : X' \rightarrow X$ be the projection. The relevant canonical relation is now $\Gamma_{p_{X'} \times p_{X'}}$ in (19), and it is naturally diffeomorphic to the horizontal cotangent bundle of $p_{X'} \times p_{X'}$. The fibre product $F_{p_{X'} \times p_{X'}}$ of Σ' and $\Gamma_{p_{X'} \times p_{X'}}$ is naturally diffeomorphic to Σ' . The projection $p_{F_{p_{X'} \times p_{X'}}} : F_{p_{X'} \times p_{X'}} \cong \Sigma' \rightarrow \Gamma_{p_{X'} \times p_{X'}} \circ \Sigma' = \Sigma_0$ is now the quotient map under the G -action.

Let $q' : T^*(X_0 \times X_0) \times T^*(X' \times X') \rightarrow T^*(X' \times X')$ be the projection onto the second factor. Fix for now $\zeta' = (x, r\alpha'_x, x, -r\alpha'_x) \in \Sigma'$; the differential of q' at (ζ_0, ζ') ,

$$q'_{\zeta'} =: d_{(\zeta_0, \zeta')} q : T_{\zeta_0}(T^*(X_0 \times X_0)) \times T_{\zeta'}(T^*(X' \times X')) \rightarrow T_{\zeta'}(T^*(X' \times X')),$$

is simply projection onto the second factor. To ease the exposition, we shall now assume that the local coordinates on the open neighbourhood $U \subseteq X'$ of x have been so chosen that the vector fields ξ^\sharp are constant (since the action of G along $\Phi^{-1}(0)$ is free, this may certainly be done).

Lemma 25. *In this system of local coordinates, we have*

$$\text{Jac}_y(\alpha')\xi^\sharp(y) = \text{Jac}_y(\alpha')^t\xi^\sharp(y) = 0,$$

for any $y \in U$, $\xi \in \mathfrak{g}$.

Proof. The two vanishings are equivalent, since the vector fields ξ^\sharp , $\xi \in \mathfrak{g}$, span the kernel of $d\alpha'$. Since in addition the ξ^\sharp 's are horizontal on X' , the asserted vanishings follow by differentiating the equality $\alpha'(\xi^\sharp) = 0$.

In these local coordinates,

$$q'_{\zeta'}(T_{(\zeta_0, \zeta')}(\Gamma_{J \times J})) = \left\{ \begin{pmatrix} v \\ \phi \\ w \\ \eta \end{pmatrix} : v, w, \phi, \eta \in \mathbb{R}^{2n-g+1} \text{ such that } \right. \\ \left. \phi(\xi^\sharp(x)) = \eta(\xi^\sharp(x)) = 0, \forall \xi \in \mathfrak{g} \right\}.$$

The symplectic annihilator in $T_{\zeta'}(T^*(X) \times T^*(X))$ is then

$$q_{\zeta'}(T_{(\zeta', \zeta)}(\Gamma_{J \times J}))^\perp = \left\{ \begin{pmatrix} \xi^\sharp(x) \\ 0 \\ \eta^\sharp(x) \\ 0 \end{pmatrix} : \xi, \eta \in \mathfrak{g} \right\}.$$

We shall now denote by V_0 , V_1 and V the analogues in the present setting of the vector bundles U_0 , U_1 and U introduced in section 3. In view of Lemma 25 we now have

$$V_{0\zeta'} = \left\{ \begin{pmatrix} \xi^\sharp(x) \\ 0 \\ \xi^\sharp(x) \\ 0 \end{pmatrix} : \xi \in \mathfrak{g} \right\} \text{ and } V_{1\zeta} = \left\{ \begin{pmatrix} \xi^\sharp(x) \\ 0 \\ \eta^\sharp(x) \\ 0 \end{pmatrix} : \xi, \eta \in \mathfrak{g} \right\}.$$

Hence, in the terminology of Corollary 7 iii) and recalling Corollary 9, we obtain for the quotient bundle $V = V_1/V_0 \subseteq N_{\Sigma'} \cong N_U$:

$$V_{\zeta'} = \left\{ \begin{pmatrix} \xi^\sharp(x) \\ 0 \\ -\xi^\sharp(x) \\ 0 \end{pmatrix} : \xi \in \mathfrak{g} \right\} \cong \mathfrak{g}_r^M(p) \subseteq N_U.$$

Again by Corollary 7, the symplectic normal bundle N_V of V in $N_{\Sigma'}$ is thus

$$N_V \cong (q^*H(M'/M_0), 2r\Omega_{H(M/M')}) \oplus (q^*H(M'/M_0), -2r\Omega_{H(M/M')}).$$

This is clearly symplectically isomorphic to the normal bundle of Σ_0 in $T^*(X_0 \times X_0)$ (cfr Proposition 6.4 of [BG]). By Corollary 7 v), the image of $\sigma_{\mathfrak{g}^M, J} \otimes \sigma_{J_0}$ under the vector bundle morphism $\Phi_V : \mathcal{S}(N_{\Sigma'}) \rightarrow \bigwedge^{-1/2}(V) \otimes \mathcal{S}(N_V)$ is

$$\Phi_V(\sigma_{\mathfrak{g}^M, J} \otimes \sigma_{J_0}) = \text{vol}_V^{-1/2} \otimes \sigma_{J_0},$$

where $\text{vol}_V^{-1/2}$ is the $-\frac{1}{2}$ -form on the oriented isotropic subbundle $V \subseteq T(T^*(X' \times X'))$ taking value one on oriented orthonormal basis of V .

The following Lemma is proved with an argument similar to that used for Lemma 9, the main difference being we need to include an oriented orthonormal basis for $V_{0, \zeta'}$.

Lemma 26. *Suppose $x \in X'$ and let $\zeta' = (x, \alpha'_x, x, -\alpha'_x)$ (thus, $r = 1$). Then the image at ζ' of $\text{vol}_{\Gamma_{p_{X'} \times p_{X'}, \zeta'}}^{1/2} \otimes \text{vol}_{\Sigma', \zeta'}^{1/2} \otimes \text{vol}_{V, \zeta'}^{-1/2}$ in $\bigwedge^{1/2}(\Sigma_0)_{\zeta_0} \otimes \det(V_{0, \zeta'}^*)$ under the line bundle isomorphism (14) is $2^{-g} \text{vol}_{\Sigma_0, \zeta_0}^{1/2} \otimes \text{vol}_{V_{0, \zeta'}}$, where vol_{V_0} is the volume form on V_0 taking value one on oriented orthonormal frames.*

Clearly, $\text{vol}_{V_{0, \zeta'}}$ is the volume form on the vertical tangent bundle of the principal G -bundle $\Sigma' \rightarrow \Sigma_0$ associated to the orientation and the restricted metric. The statement of Theorem 2 then follows by fibrewise integration and homogenization.

8 The Asymptotic Expansion

The function $\sum_{j=1}^{d_k} s_j^{(G, k)}(x') \otimes \overline{s_j^{(G, k)}(y')}$ appearing in (16) is obviously well-defined on X_0 . We shall now argue that it admits an asymptotic expansion as in the statement of Theorem 1.

To this end, let us introduce the following auxiliary Fourier-Hermite distribution on $X_0 \times X_0$:

$$\begin{aligned} \tilde{P}_{X|X_0} &=: (V_{\text{eff}} \boxtimes V_{\text{eff}})^{-1} \tilde{\Pi}_{X|X_0} \\ &= \left(\sum_{k=0}^{+\infty} \sum_{j=1}^{d_k} s_j^{(G, k)} \boxtimes \overline{s_j^{(G, k)}} \right) \text{vol}_{X_0}^{1/2} \boxtimes \text{vol}_{X_0}^{1/2}. \end{aligned} \quad (40)$$

Thus $\tilde{P}_{X|X_0} \in J^{(1+g)/2}(X_0 \times X_0, \Sigma_0)$, and the associated operator $P_{X|X_0} : \mathcal{D}'_{1/2}(X_0) \rightarrow \mathcal{D}'_{1/2}(X_0)$ is an S^1 -invariant elliptic operator satisfying $\Pi_{X_0} \circ P_{X|X_0} = P_{X|X_0}$. One can now follow the arguments in [STZ], Lemma 4.2 and 4.3 and [G] to conclude that $P_{X|X_0}$, and thus $\Pi_{X|X_0}$, is an elliptic Toeplitz operator possessing a semiclassical symbol, and that this implies the asserted asymptotic expansion.

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